SECTION «PHYSICAL AND MATHEMATICAL SCIENCES»

PHYSICAL AND MATHEMATICAL FOUNDATIONS OF HEAT-TRANSFER MODELING IN MULTILAYER SYSTEMS WITH THERMAL CONTACT RESISTANCE

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Abstract. In this study, approaches to the physical and mathematical modeling of nonlinear heat transfer in multilayer systems are systematically investigated, a topic that has become particularly relevant in the context of growing demands for thermal resistance, energy efficiency, and reliable prediction of transient temperature fields in heterogeneous protective structures and layered media. It is demonstrated that traditional linear and steady-state thermal models do not provide an adequate level of fidelity for analyzing real multilayer systems, since they neglect the temperature dependence of thermal properties, the influence of interlayer thermal contact resistance, and the transient character of external thermal loading.

The study substantiates the necessity of transitioning from simplified linear formulations to integrated nonlinear transient models, within which a multilayer structure is considered as a spatially heterogeneous medium with distributed conductive, inertial, and barrier properties. Particular

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attention is paid to the combination of classical heat-conduction theory with temperature-dependent thermal conductivity and volumetric heat capacity, Robin-type boundary conditions, interfacial conditions with thermal contact resistance, and scenario-based external thermal loading.

A key element of the proposed approach is the interpretation of the multilayer system not as a passive sequence of layers with fixed parameters, but as a nonlinear transient medium whose thermal response evolves under the combined influence of material heterogeneity, interlayer interaction, and time-varying boundary conditions. This makes it possible to consistently describe the delay of the thermal wave, the smoothing of impulse thermal loads, the redistribution of temperature gradients across the thickness, and the influence of imperfect thermal contact on the overall barrier behavior of the system.

The subject of the research encompasses the processes of modeling, analysis, verification, and scenario-based assessment of transient temperature fields in multilayer systems with temperature-dependent thermal properties and interlayer thermal contact resistance. Within the framework of the study, classical and nonlinear heat-conduction models, canonical nonlinear diffusion equations, analytical and semi-analytical benchmark solutions, implicit finite-difference schemes, Picard-type iterative procedures, as well as the software implementation of the proposed model for numerical scenario analysis are considered.

The research methodology is based on an integrated approach that combines classical heat-transfer theory, nonlinear mathematical physics, transient finite-difference modeling, benchmark-based verification, elements of group-theoretical interpretation of canonical heat equations, and computational scenario analysis. At the first stage, contemporary scientific approaches to multilayer heat conduction, nonlinear thermal transport, interfacial heat-transfer modeling, and analytical or numerical solution methods are systematized. At the second stage, an original nonlinear transient mathematical model is developed, incorporating multilayer geometry, temperature-dependent coefficients, interlayer contact resistance, Robin boundary conditions, and scenario-based thermal loading. At the third stage, the model is implemented in Python, and numerical simulations are performed for steady, cyclic, pulse, and localized high-intensity thermal regimes.

The aim of the research is to develop a scientifically grounded physical and mathematical model of nonlinear transient heat transfer in multilayer systems that enables the analysis of temperature fields under conditions of heterogeneous structure, temperature-dependent thermal properties, interlayer thermal resistance, and variable thermal loading. Achieving this objective involves integrating classical heat-conduction formulations with nonlinear constitutive relations, constructing a consistent multilayer boundary-value problem, numerically validating the proposed approach, and interpreting the obtained results in terms of thermal inertia, barrier efficiency, heat-flux attenuation, and delay of the thermal response. The implementation of such an approach is aimed at improving the reliability and physical adequacy of predictive thermal analysis for multilayer protective structures and related heterogeneous heat-conducting systems.

1. INTRODUCTION

The current stage of research in heat transfer is characterized not only by increasing requirements for the energy efficiency and thermal resistance of enclosure systems, but also by a substantial increase in the complexity of the physical formulations used to analyze such systems. For multilayer protective structures, this issue is particularly important because their performance is governed not by a single isolated characteristic but by the combined effects of thermal conductivity, heat capacity, thermal inertia, interlayer interaction, and the nature of the external thermal loading. Under real operating conditions, a structure must not only reduce the instantaneous heat flux but also delay thermal-wave penetration, smooth short-term peak effects, and limit heating of the inner surface, which directly determines the thermal state of the internal space [4, pp. 1–10; 17, pp. 3373–3391].

Under these conditions, the traditional treatment of an enclosing structure as a homogeneous body with constant thermophysical properties is insufficient. Classical linear models describe steady heat transfer and simplified transient regimes well; however, in multilayer systems with temperature-dependent properties they face substantial limitations. The most important of these are the inability to adequately account for changes in thermal conductivity and heat capacity with increasing temperature, the difficulty of describing interlayer interfaces, and insufficient sensitivity to the actual dynamics of short-duration thermal

pulses. Consequently, contemporary research is gradually shifting from simplified steady-state and linear formulations toward nonlinear transient models of multilayer media that combine classical heat-conduction theory, variable transport coefficients, thermal contact resistances, and numerical methods [5, pp. 1001–1009; 7, pp. 45–58].

The temperature dependence of thermophysical properties is of particular importance. In the simplest engineering treatment, thermal conductivity and effective heat capacity may be assumed constant, which considerably simplifies the mathematical formulation. For real multilayer systems, however, especially those containing composite, recycled, or functionally heterogeneous materials, this assumption often loses physical adequacy. As temperature changes, not only the intensity of heat transfer but also the ability of the medium to store heat changes, making the problem intrinsically nonlinear. This circumstance requires a transition from classical schemes to quasilinear heat-conduction equations in which the model coefficients depend on the unknown temperature field [7, pp. 45–58; 18, pp. 1–8].

Accounting for interlayer interaction is equally important. In classical multilayer models, interfaces between materials are often treated as ideal, implying continuity of both temperature and heat flux. In real structures, however, the material interface itself may become an additional thermal barrier when contact between adjacent layers is imperfect. In this case, thermal contact resistance arises, producing a temperature jump at the interface and altering the overall thermal behavior of the system. Modern approaches to multilayer heat-transfer modeling therefore increasingly employ formulations in which interlayer contacts are treated not as a secondary technical refinement but as a fully fledged physical component of the model [5, pp. 1001–1009; 9, pp. 1–8].

An important step in the development of this field has been the integration of multilayer thermal models with analytical, semi-analytical, and numerical methods. Analytical solutions remain valuable as benchmark references for verifying computational algorithms and as a means of qualitatively interpreting individual regimes. At the same time, as the model is extended to nonlinear heat conduction, contact resistances, and transient boundary conditions, numerical schemes capable of reproducing the spatiotemporal evolution of the temperature field without loss of physical meaning become increasingly important. The subsequent development of the field

naturally led to the use of implicit difference schemes, Picard-type iterative procedures, and dedicated verification tests in which numerical solutions are compared with benchmark analytical or semi-analytical regimes [2, pp. 192–213; 16, pp. 3175–3187; 20, pp. 1279–1296].

At the same time, an analysis of contemporary studies shows that a substantial proportion of existing solutions either focus on individual aspects of the problem or are restricted to simplified configurations of the medium. Classical multilayer models provide a physically transparent basis but often account insufficiently for temperature nonlinearity. Nonlinear formulations reproduce coefficient variability well but are frequently considered outside a complete multilayer structure or without sufficient attention to contact resistance. Numerical methods offer substantial flexibility, yet require explicit verification and physically motivated interpretation of the results. Consequently, there remains a need for a consistent model that simultaneously combines multilayer geometry, temperature-dependent thermophysical parameters, interlayer interaction, scenario-based external effects, and a verified numerical implementation [7, pp. 45–58; 19, pp. 1–8].

The theoretical aspect of the problem is equally important. A complete applied model of a multilayer structure with variable coefficients, contact resistances, and transient boundary conditions is too complex for a direct analytical solution in general form. The canonical class of nonlinear heat-conduction equations is therefore of particular value because it allows the applied formulation to be interpreted in the broader context of mathematical physics. In this sense, group-theoretical analysis plays not merely an auxiliary but a structuring role: it does not replace the complete numerical model, but relates it to classes of symmetries, reductions, and exact solutions known for canonical forms of nonlinear diffusion [11, pp. 347–358; 14, pp. 1–6].

Within the present study, the key scientific problem is the development of a consistent heat-transfer model for a multilayer medium that combines classical heat-conduction relations, a temperature-dependent description of thermophysical properties, interface conditions with thermal contact resistance, scenario-based temporal variation of thermal loading, and tools for numerical verification and theoretical interpretation. In other words, the objective is to bridge the gap between classical linear schemes, nonlinear quasilinear formulations, and applied tools for the numerical analysis of

multilayer structures. This gap, clearly identifiable in the contemporary literature, defines the direction of the present research.

The aim of this study is to substantiate the theoretical foundations and develop an applied nonlinear transient heat-transfer model for a multilayer structure that enables temperature fields to be analyzed under temperature-dependent thermophysical properties, interlayer thermal contact resistance, and variable external thermal effects. Achieving this aim requires solving a set of interrelated tasks: analyzing contemporary scientific approaches; identifying the classical and canonical principles required to construct the proposed model; formalizing the nonlinear transient multilayer formulation; developing a numerical scheme for its implementation; verifying the computational algorithm; and performing numerical investigations of physically meaningful thermal-loading scenarios.

2. Analysis of Contemporary Scientific Approaches

2.1. Classical Heat-Conduction Models in Homogeneous and Multilayer Media

The classical stage in the development of heat-conduction theory is associated with Fourier's law and the linear heat equation, which laid the foundation for describing steady and transient temperature fields in continuous media. For homogeneous bodies, these models reproduce temperature distributions with sufficient accuracy when thermophysical properties are constant and the boundary conditions are simple. They therefore became the starting point for subsequent extensions to more complex problems of mathematical physics [1, pp. 1–12; 16, pp. 3175–3187].

Subsequent developments were associated with multilayer systems, in which the thermal process is determined not only by the external loading but also by the sequential arrangement of layers with different thermal conductivities and heat capacities. For such media, a steady-state formulation is convenient as a first approximation, but in real structures it does not represent thermal inertia, delayed thermal-wave penetration, or the smoothing of short-duration disturbances. Contemporary studies therefore increasingly employ transient multilayer heat-conduction models in which temperature is treated as a function of space and time, while the internal structure of the medium is represented through layerwise variations of properties across the thickness [3, pp. 1–10; 8, pp. 1–8].

For a multilayer system, the through-thickness profile of thermophysical properties is particularly important because it determines which layers primarily conduct heat, which act as inertial heat-storage elements, and which form thermal-insulation barriers (Figure 2.1). In this sense, a multilayer system is not merely a “wall” but a physically heterogeneous medium with distributed barrier and storage properties.

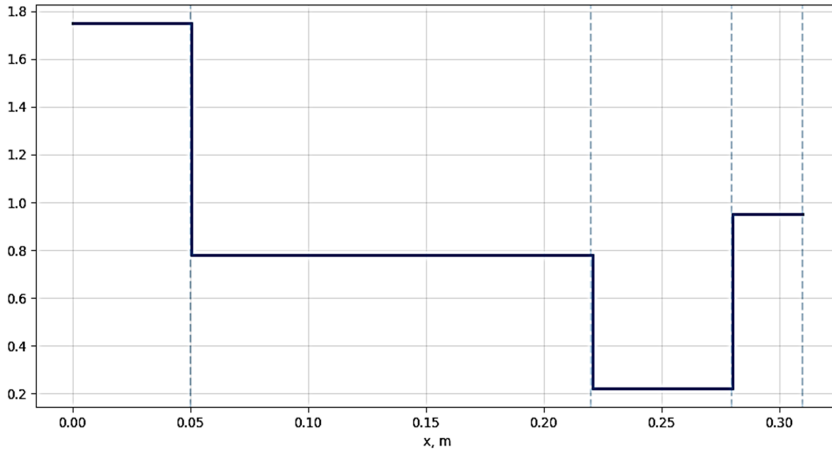


Figure 2.1. Baseline thermal-conductivity profile $\lambda_0(x)$ of the multilayer medium

2.2. Nonlinear Heat-Transfer Models with Temperature-Dependent Coefficients

The next stage of development is associated with abandoning the assumption of constant thermophysical properties. For many real materials, especially composites, recycled materials, and specially modified materials, thermal conductivity and effective volumetric heat capacity vary with temperature. In this case, the heat-conduction equation becomes nonlinear because the diffusion coefficient itself depends on the unknown temperature field [7, pp. 45–58; 18, pp. 1–8].

The physical meaning of this nonlinearity is that the ability of the medium to conduct and store heat changes together with its thermal state. Consequently, even for identical geometry, the response of a structure

to an external thermal effect may differ substantially depending on the temperature level, loading duration, and material properties. Nonlinear models therefore provide a substantially more adequate description of transient processes in multilayer media than their linear counterparts [4, pp. 1–10; 19, pp. 1–8].

The role of thermal diffusivity $a = \lambda / (\rho c)$, which effectively determines the rate at which the system responds to a thermal disturbance, should be emphasized separately. Whereas thermal conductivity characterizes the ability to transfer heat and heat capacity characterizes the ability to store it, thermal diffusivity combines both effects into a single dynamic parameter. Consequently, in transient problems it is useful to analyze not only $\lambda(T)$ (Figure 2.2) but also the spatial and temperature dependence of $a(x,T)$, (Figure 2.3) [7, pp. 45–58; 18, pp. 1–8].

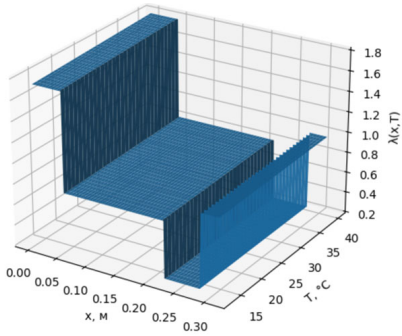


Figure 2.2. Spatial and temperature-dependent distribution λ(x,T)

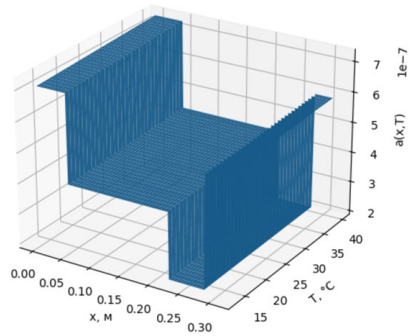


Figure 2.3. Spatial and temperature-dependent distribution of thermal diffusivity a(x,T)

2.3. Approaches to Modeling Thermal Contact between Layers

In multilayer systems, it is essential not only to describe the individual layers but also to account correctly for the interfaces between them. In the simplest formulation, the contact between layers is assumed ideal, so that temperature and heat flux are continuous across the interface. This scheme is mathematically convenient, but in real structures it is often overly simplified [5, pp. 1001–1009; 9, pp. 1–8].

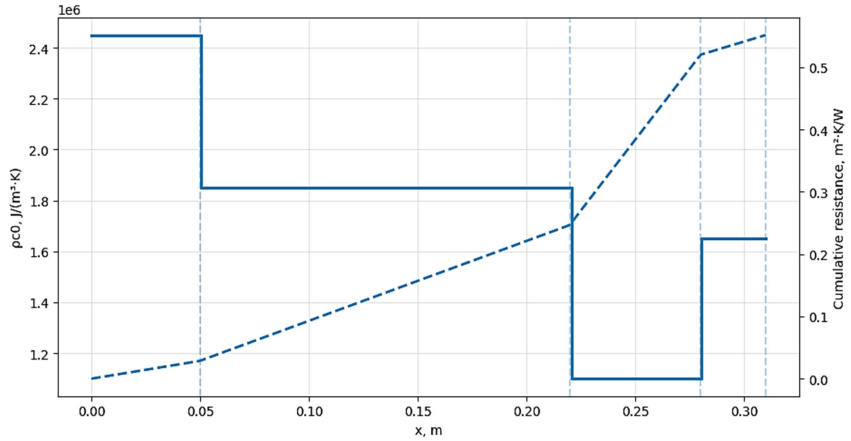


Figure 2.4. Thermal inertia of the layers and cumulative thermal resistance across the structural thickness

When surface roughness, interlayers, insufficient contact pressure, or pronounced differences between adjacent materials are present, thermal contact resistance arises. The heat flux remains continuous across the interface, but the temperature may exhibit a jump. Physically, this means that the interface itself acts as an additional barrier to heat transfer (Figure 2.4). Accounting for such effects is therefore a characteristic feature of modern multilayer heat-conduction models, particularly in problems where not only the steady thermal state but also the heating dynamics are important [9, pp. 1–8; 10, pp. 1–9].

2.4. Analytical, Numerical, and Semi-Analytical Research Methods

For simplified cases, including homogeneous or linear multilayer systems, analytical methods remain effective: separation of variables, integral transforms, series solutions, and models for bilayer and multilayer slabs. Their principal advantage is that they provide benchmark solutions suitable for verifying numerical algorithms and for qualitatively analyzing the effects of parameters [16, pp. 3175–3187; 20, pp. 1279–1296].

As model complexity increases, however, and multilayer structure, temperature-dependent coefficients, contact resistance, and transient

boundary conditions are considered simultaneously, an analytical description rapidly loses universality. Numerical methods, particularly implicit difference schemes that provide stability under demanding loading regimes, therefore become the main research tool. Nonlinearity is generally treated iteratively, for example by Picard-type procedures or related sequential coefficient updates [7, pp. 45–58; 19, pp. 1–8].

Semi-analytical verification approaches are also particularly important, with the numerical solution compared against an exact or benchmark solution for a dedicated test regime. This combination of analytical benchmarking, numerical stability, and physical interpretability is currently among the most productive approaches to transient heat-transfer problems in heterogeneous media [2, pp. 192–213; 18, pp. 1–8].

2.5. Group-Theoretical Approaches to Canonical Nonlinear Heat-Conduction Equations

Alongside direct numerical simulation, group-theoretical analysis of nonlinear heat-conduction equations occupies an important place in the contemporary literature. Its purpose is not to replace the applied multilayer model, but to identify a canonical class of equations for which symmetries, invariant substitutions, self-similar regimes, and individual exact solutions can be described [11, pp. 347–358; 14, pp. 1–6].

For equations of the form $u_t = \left(K\left(u\right)u_x\right)_x + Q\left(u\right)$, the structure of the diffusion coefficient and source term directly determines the admissible symmetries and the nature of possible reductions. The group-theoretical approach should therefore be regarded as the theoretical core for analyzing nonlinear heat conduction. At the same time, a complete multilayer boundary-value problem containing discontinuous properties, contact resistances, and complex boundary conditions is generally not a direct object of a full Lie classification. Symmetry analysis therefore performs an explanatory and structuring function here, whereas the complete applied model must ultimately be investigated numerically [2, pp. 192–213; 15, pp. 1–46].

Thus, an analysis of contemporary scientific approaches shows that heat-transfer models have evolved from classical linear formulations for homogeneous bodies to nonlinear transient models of multilayer media with temperature-dependent coefficients and interlayer thermal contact

resistance. Analytical methods retain their importance as a benchmark basis, numerical schemes are the principal tool for complete applied formulations, and group-theoretical analysis provides a generalized mathematical framework for interpreting nonlinearity [5, pp. 1001–1009; 11, pp. 347–358].

At the same time, existing approaches tend to focus either on individual aspects of multilayer structure, on nonlinearity, or on numerical implementation itself. There therefore remains a need for a consistent model that simultaneously accounts for the heterogeneous structure of the medium, temperature dependence of thermophysical properties, thermal contact resistance, transient external-loading scenarios, and numerical verification. This consideration motivates the transition in the next section to the construction of the proposed mathematical model of transient heat transfer in a multilayer system.

3. Main Part

3.1. Classical and Canonical Heat-Conduction Models as the Theoretical Basis for the Proposed Model

Unlike the previous section, the present section focuses not on a general description of contemporary approaches but on identifying those principles of classical and canonical heat-conduction theory that directly form the foundation for constructing the proposed nonlinear transient model of a multilayer structure. This transition is essential because the subsequent mathematical formulation must rely not on a mere list of published results but on a consistent set of fundamental relations that ensure physical interpretability, mathematical rigor, and suitability for numerical implementation [7, pp. 45–58; 18, pp. 1–8].

The starting point is the classical heat equation, which in the simplest one-dimensional case describes the variation of temperature $T(x,t)$ in a homogeneous medium with constant thermophysical properties. In this formulation it is a linear parabolic equation expressing the fundamental balance between heat storage and diffusive transport. This model provides the basic physical representation of thermal-disturbance propagation, permits steady and transient regimes to be defined, and establishes the theoretical framework for subsequent extensions to more complex cases [16, pp. 3175–3187; 20, pp. 1279–1296].

At the level of real enclosure structures, however, the assumption of a homogeneous medium is already excessively simplified. A multilayer system consists of consecutively arranged media with different thermal conductivities, heat capacities, and temperature sensitivities. Thus, even in a one-dimensional approximation, the temperature field in such a structure is determined not only by the external thermal loading but also by the internal structure of the system, the order and thicknesses of the layers, and the properties of their interfaces. The transition from a homogeneous body to a multilayer structure is therefore not merely a technical refinement but a qualitative extension of the physical model [1, p. 125500; 3, pp. 1–10].

In the classical multilayer formulation, a separate heat equation is written for each layer, after which the local problems are coupled by interface conditions at material boundaries. If the contact between layers is ideal, temperature and heat flux remain continuous. If thermal contact resistance is taken into account, the condition of temperature continuity is replaced by a relation involving a temperature jump proportional to the heat flux. This extension of classical theory is particularly important for real multilayer structures, where an interface between layers often acts as an independent thermal barrier [5, pp. 1001–1009; 9, pp. 1–8].

The next development of the classical theory involves accounting for the temperature dependence of thermophysical properties. If the thermal conductivity λ and effective volumetric heat capacity ρc are no longer constant but vary with temperature, the heat equation becomes nonlinear. In the most general one-dimensional case, such a model can be written as

$$(\rho c)(T),\partial T/\partial t = \partial/\partial x\left(\lambda(T),\partial T/\partial x\right) + Q(x,t,T).$$

In this expression, nonlinearity is concentrated in two terms. First, it enters the diffusion operator through the temperature-dependent thermal conductivity $\lambda(T)$. Second, it may arise in the storage term through the dependence $(\rho c)(T)$. If an internal heat source is also included, the function $Q(x,t,T)$ may introduce an additional nonlinear contribution. This extension moves the problem from the class of linear heat-conduction problems to the broader class of nonlinear parabolic boundary-value problems [7, pp. 45–58; 18, pp. 1–8].

A complete applied multilayer formulation that simultaneously includes discontinuous coefficients, thermal contact resistances, transient boundary conditions, and internal sources is, however, too complex for direct

theoretical analysis in general form. For this reason, the canonical class of nonlinear heat-conduction equations becomes particularly important

$$u_t = \left(K(u), u_x\right)_x + Q(u),$$

and is used as a generalized mathematical core for the theoretical interpretation of the complete model. Its advantage is that, on the one hand, it retains the principal features of nonlinear heat transfer and, on the other, has a sufficiently compact structure for symmetry analysis, construction of invariant reductions, and identification of typical solution classes [11, pp. 347–358; 14, pp. 1–6].

Several fundamentally important cases can be distinguished within this canonical class. If $K(u) = \text{const}$ and $Q(u) = 0$, the classical linear heat equation is recovered. If $K(u)$ is a power-law function, a nonlinear power-law class arises. If $K(u)$ has an exponential form, another type of nonlinear diffusion characteristic of temperature-dependent coefficients is obtained. If the source term $Q(u)$ is nonzero, the model combines nonlinear transport with local heat generation. The same logic is also used in the software implementation, where the multilayer model is heuristically reduced to an effective canonical representative of the class $u_t = \left(k(u)u_x\right)_x + Q(u)$, after which the type of nonlinearity and the possible symmetries and reductions are interpreted.

In this context, the group-theoretical block is not a separate “add-on” to the model and does not claim to provide a complete classification of the entire multilayer boundary-value problem. Its role is different: it explains which type of nonlinear parabolic equation the constructed model approaches, which canonical regimes may be associated with its simplified forms, and why the complete applied problem requires numerical rather than purely analytical investigation. This is particularly important because it allows the applied model not only to be described but also to be embedded in the broader context of mathematical physics [11, pp. 347–358; 15, pp. 1–46].

Thus, the theoretical basis of the subsequent formulation consists of four key principles. First, transient heat transfer in a real structure should be treated as heat diffusion in a heterogeneous medium. Second, multilayer structure must be incorporated into the model not only through material parameters but also through interface conditions at layer boundaries. Third, the temperature dependence of $\lambda(T)$ and $(\rho c)(T)$ is fundamental

and cannot be reduced to a constant correction without loss of physical meaning. Fourth, the complete applied problem is naturally associated with the canonical class of nonlinear heat-conduction equations, which serves as its theoretical core but does not replace the full numerical formulation. These principles determine the logic of the next subsection, in which the proposed nonlinear transient heat-transfer model for a multilayer structure is formulated [7, pp. 45–58; 14, pp. 1–6].

3.2. Development of a Nonlinear Transient Heat-Transfer Model for a Multilayer Structure

Based on the classical and canonical principles outlined above, we formulate the proposed mathematical model of transient heat transfer in a multilayer structure. Its construction is oriented not toward an abstract theoretical class but toward a physically meaningful system that preserves mathematical rigor and remains suitable for numerical implementation. The model is based on a plane multilayer enclosure of total thickness L, formed by consecutively arranged layers with different thermophysical properties. The spatial coordinate x is measured from the outer to the inner surface, and the j th layer occupies the interval $x_{j-1} \le x \le x_j$, where $j = 1,\ldots,N$ [3, pp. 1–10; 7, pp. 45–58].

Physically, the structure is treated as a heterogeneous multilayer medium in which the outer layer performs protective and partly load-bearing functions, intermediate layers may provide thermal storage or thermal insulation, and the inner layer directly adjoins the internal space. Each layer is assumed macroscopically homogeneous within its own thickness, while its thermal conductivity and effective volumetric heat capacity are allowed to depend on temperature. This assumption brings the model into agreement with the actual behavior of structures containing unconventional, composite, or resource-efficient materials [4, pp. 1–10; 21].

The principal geometrical assumption is one-dimensional heat transfer. It is justified when the in-plane dimensions of the structure are much greater than its thickness and the thermal effect over the region under consideration can be regarded as uniform across the surface. Under these conditions, the dominant heat flux is directed normal to the wall, and two-dimensional edge effects may be neglected in a first approximation. This assumption does not deny the existence of more complex spatial regimes but provides

a reasonable compromise between physical adequacy and mathematical tractability [3, pp. 1–10; 16, pp. 3175–3187].

For each layer, the local transient heat-transfer equation is written in the form

$$(\rho c)\, j(T), \partial T_j / \partial t = \partial / \partial x\left(\lambda_j(T), \partial T_j / \partial x\right) + \\ + Q_j(x,t,T), \quad xj-1 < x < x_j.$$

Here $T_j(x,t)$ is the temperature in the jth layer, $\lambda_j(T)$ is the temperature-dependent thermal conductivity of the layer material, $(\rho c)_j(T)$ is its effective volumetric heat capacity, and $Q_j(x,t,T)$ is the volumetric heat source. In the general case the source term may be absent, but its inclusion is essential for scenarios involving a localized thermal pulse or internal heating [18, pp. 1–8; 19, pp. 1–8].

In the software implementation, the temperature dependence of the coefficients is specified in a linearized form relative to the reference temperature T_{ref} :

$$\lambda_j(T) = k_{0j}\left(1 + \beta_{k,j}\left(T - T_{ref}\right)\right),$$
$$(\rho c)\, j(T) = \rho c_{0j}\left(1 + \beta_{rc,j}\left(T - T_{ref}\right)\right).$$

This representation is sufficiently flexible to describe moderate nonlinearity characteristic of a wide range of construction and composite materials while remaining straightforward to implement numerically. It does not claim to be a universal thermodynamic law for all materials, but it is a justified approximation that preserves the principal physical meaning: both the intensity of heat conduction and the capacity of the medium to store heat vary with temperature. This dependence places the model within the class of quasilinear nonlinear heat-conduction equations [7, pp. 45–58; 18, pp. 1–8]. The same form is directly implemented in the code through the conductivity() and volumetric_heat_capacity() methods, with each layer characterized by its own parameters k_0, β_k, ρc_0, and β_{rc}.

The initial state of the structure is specified by the condition

$$T_j(x,0) = T_0(x),$$

where the function $T_0(x)$ defines the initial temperature distribution through the thickness of the system. In the simplest scenario, it may be constant and correspond to a uniformly heated or uniformly cooled structure. The general form of the initial condition also permits a spatially nonuniform

initial state when this is required to model the preceding thermal history of the structure [16, pp. 3175–3187].

Robin boundary conditions accounting for convective heat exchange with the surroundings are imposed on the outer and inner surfaces. For the outer surface at $x = 0$, we have

$$-\lambda_J(T_1)\frac{\partial T_1}{\partial x} = h_{out}\left[T_{out}(t) - T_s\right]$$

and for the inner surface at $x = L$

$$-\lambda_N(T_N)\frac{\partial T_N}{\partial x} = h_{in}\left[T_s - T_{in}(t)\right]$$

with the adopted heat-flux direction taken into account.

Here h_{out} and h_{in} are the heat-transfer coefficients at the outer and inner surfaces, while $T_{out}(t)$ and $T_{in}(t)$ are the temperatures of the corresponding environments. The use of Robin conditions is physically motivated because it allows the model to move beyond simplified formulations with prescribed surface temperature toward a formulation in which heat flux is determined by the difference between surface and ambient temperatures [18, pp. 1–8; 20, pp. 1279–1296].

In the program code, this concept is implemented directly in the assembly procedure for the discrete system, where effective heat-transfer coefficients are imposed at the boundary nodes for the outer and inner surfaces. In addition, the outer surface optionally includes linearized radiative heat exchange through an additional radiative contribution to the heat-transfer coefficient. Thus, the boundary conditions implemented in the model extend beyond a purely convective scheme and permit refinement of the external heat exchange.

At the interfaces between adjacent layers $x = x_j$, the interface conditions are specified in two different ways depending on whether the contact is assumed ideal. For ideal contact, continuity of temperature and heat flux is imposed:

$$T_j(x_j, t) = T_{j+1}(x_j, t),$$
$$\lambda_j(T), \partial T_j / \partial x = \lambda_{j+1}(T), \partial T_{j+1} / \partial x.$$

This formulation corresponds to a mathematically idealized case in which the material interface does not create an additional thermal barrier [3, pp. 1–10; 5, pp. 1001–1009].

If thermal contact resistance $R_{c,j}$ is present between the layers, temperature continuity is violated and is replaced by the relation

$$T_j\left(x_j,t\right)-T_{j+1}\left(x_j,t\right)=q_j\left(t\right),R_{c,j},$$

where $q_j\left(t\right)$ is the heat flux through the corresponding interface. Physically, this means that, for the same heat flux, the interface creates an additional temperature drop and therefore affects both the total thermal resistance of the structure and the shape of the temperature profile [9, pp. 1–8; 10, pp. 1–9].

In the numerical implementation, thermal contact resistances are incorporated through an effective conductance at grid half-steps. If there is no material interface between adjacent nodes, harmonic averaging of the conductivities is used. If the two nodes belong to different layers, the contribution of thermal contact resistance is added to the equivalent conductance. This approach is convenient because it preserves a unified discretization logic throughout the grid while enabling the temperature jump at an interface to be modeled correctly. Thus, the numerical scheme does not merely "account for" contact resistance but incorporates it as a fully fledged element of the model's internal structure.

The scenario-based nature of the external thermal loading plays a separate role in the model. Unlike a steady-state formulation with constant external temperature, the function $T_{out}\left(t\right)$ can describe different regimes: a constant thermal background, daily fluctuations, a short-duration thermal pulse, or a localized high-intensity pulse. The software implementation therefore includes separate scenario classes: ConstantScenario, DailyCycleScenario, ThermalPulseScenario, and LocalizedHeatPulseScenario. Accordingly, the model is not restricted to a single loading regime and can be used to investigate the thermal behavior of the structure over a broad range of transient thermal-loading regimes.

For a localized thermal pulse, an internal volumetric heat source is additionally introduced into the model; it is active only within a prescribed portion of the wall thickness and over a limited time interval. This extension is important because it enables not only time-varying external temperatures but also local thermal disturbances acting within part of the medium to be modeled. Mathematically, the complete model therefore admits both nonautonomous boundary conditions and internal spatiotemporal heterogeneity of source type [18, pp. 1–8; 19, pp. 1–8].

To generalize the results and identify the governing regime characteristics, it is useful to rewrite the problem in dimensionless form. We introduce the characteristic length L, characteristic temperature difference ΔT, and characteristic time t_c. The dimensionless variables may be written as

$$\xi = x / L, \quad \tau = t / t_c, \quad \theta = \left(T - T_{ref}\right) / \Delta T.$$

In this formulation, the equation remains nonlinear but allows the main dimensionless parameters to be identified. The Fourier number occupies a central position among them

$$Fo = \alpha_{ref}, t_c / L^2,$$

which characterizes the relative rate of thermal-disturbance penetration, together with the Biot numbers for the outer and inner surfaces

$$Bi_{out} = h_{out} L / \lambda_{ref}, \quad Bi_{in} = h_{in} L / \lambda_{ref}.$$

In the software implementation, these parameters are calculated directly in the simulation report, providing the results with not only applied but also generalized physical meaning. This is particularly important for comparing different structural configurations and loading scenarios because the results can then be interpreted not only in absolute quantities but also in a dimensionless regime space.

On this basis, several key metrics can be formulated for evaluating structural performance. First, the maximum inner-surface temperature T_{max}^{in} characterizes the peak thermal load on the internal space. Second, the time required to reach a prescribed temperature threshold reflects the ability of the structure to delay penetration of the thermal wave. Third, the heat flux at the inner surface and an integral thermal index characterize the cumulative thermal loading over the entire observation interval. All of these quantities are implemented directly in the reporting module, where the maximum and minimum inner-surface temperatures, heat fluxes, threshold-reaching time, and the integral thermal index in Wh / m^2 are calculated.

Thus, the mathematical formulation combines several levels of description. At the local level, it accounts for temperature-dependent thermal conductivity and heat capacity within each layer. At the interface level, thermal contact resistances between layers are included. At the boundary level, transient Robin conditions are imposed on the outer and inner surfaces. At the scenario level, different temporal regimes of external forcing and localized internal heat sources are permitted. This multilevel structure makes the model suitable for describing both quasi-steady regimes and short-duration thermal pulses that substantially alter the dynamics of the

temperature field. The physical property profiles illustrating heterogeneity and temperature dependence have already been presented in Figures 2.1–2.4; in the present subsection the emphasis shifts from parameter interpretation to the complete formalization of the proposed model [7, pp. 45–58; 18, pp. 1–8].

3.3. Numerical Implementation and Verification of the Model

The nonlinear transient model formulated above is physically meaningful, but in its complete form it does not admit a universal analytical solution for an arbitrary number of layers, temperature-dependent coefficients, contact resistances, and transient boundary conditions. Its investigation therefore requires a numerical implementation. In the present work, this implementation was developed in Python as a modular software package combining the mathematical core, thermal-loading scenarios, a verification block, visualization tools, and a user interface. The internal architecture separates the material library, layer models, boundary-condition and scenario blocks, numerical solver, reporting, verification, and group-analysis modules. This structure provides not merely a calculation tool for a specific regime but an integrated environment for investigating the behavior of a multilayer system.

A uniform grid along the coordinate x is used for the spatial discretization of the structure. Each grid node is assigned the number of the layer in which it is located, while layer boundaries are represented as dedicated interface points. This construction makes it possible, on the one hand, to maintain a single array of temperature values across the entire thickness of the system and, on the other hand, to account correctly for property changes when passing from one layer to another. In the software implementation, this logic is embedded in the MeshData structure, which stores the node coordinates, grid spacing, layer identifiers, and interface list simultaneously.

The interface shown in Figure 3.1 provides a convenient integration of problem formulation, specification of layer parameters, selection of the thermal-loading scenario, and subsequent analysis of the results. Its structure reflects the overall logic of the computational process: from defining the multilayer configuration to obtaining temperature profiles, heat fluxes, integral indicators, and verification characteristics. In this sense, the interface performs not only a service function but also a methodological one, because it makes the sequence of the numerical study transparent and coherent.

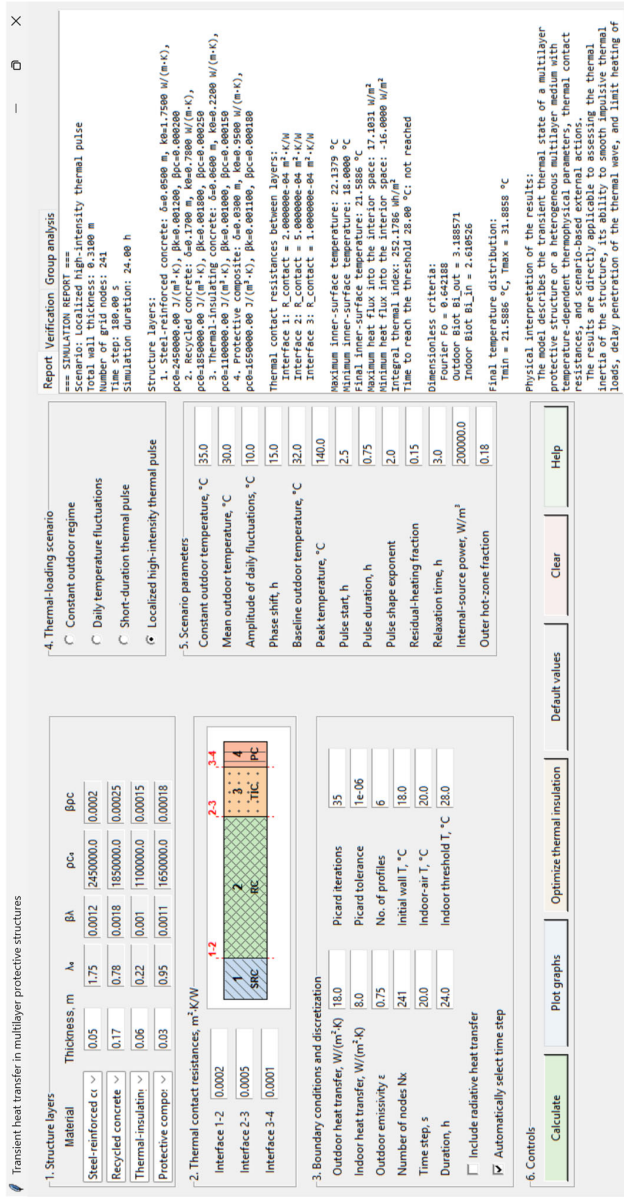


Figure 3.1. Software interface for the transient heat-transfer model of multilayer protective structures

The verification and group-theoretical analysis results shown in Figure 3.2 reflect the internal consistency and mathematical interpretability of the constructed model. The verification block includes checks of the physical correctness of the solution, analysis of behavior at interfaces between layers, and comparison of numerical results with quasi-steady and benchmark estimates. This makes it possible to assess the accuracy and reliability of the numerical implementation. At the same time, the group-analysis block relates the applied multilayer formulation to the canonical class of nonlinear heat-conduction equations, allowing the obtained results to be interpreted in the broader context of mathematical physics.

Time discretization is based on the implicit Backward Euler scheme. This choice is motivated by the fact that, for heat-conduction problems with strongly contrasting layer properties and for regimes involving intense thermal pulses, explicit schemes impose very restrictive constraints on the time step. The implicit scheme provides substantially better stability and permits physically meaningful time steps to be used without compromising the correctness of the computation [8, pp. 1–8; 19, pp. 1–8].

After discretization, the complete equation is transformed into a system of algebraic equations for the nodal temperatures at the new time level. Because of the temperature dependence of $\lambda(T)$ and $(\rho c)(T)$, however, this system remains nonlinear. It is solved using a Picard-type iterative procedure. At each iteration, the equation coefficients are evaluated from the current approximation to the temperature field; the resulting linearized system is then solved, and the process is repeated until the prescribed convergence criterion is satisfied. This scheme is natural for quasilinear problems and is consistent with the process physics: the model coefficients are successively updated in accordance with the updated thermal state of the structure [7, pp. 45–58; 18, pp. 1–8].

At internal nodes, the discretized system expresses the balance between heat storage and diffusive exchange with neighboring cells. Discrete analogues of the Robin conditions are imposed at the system boundaries, while at interfaces between layers either harmonic averaging of conductivities or an equivalent contribution from thermal contact resistance is used. The latter is particularly important because it avoids introducing a separate complicated condition for each interface and instead incorporates contact resistance directly into the global conductance matrix. Through this

procedure, the complete applied formulation retains a unified numerical implementation despite the presence of physically heterogeneous regions and contact interfaces [5, pp. 1001–1009; 9, pp. 1–8].

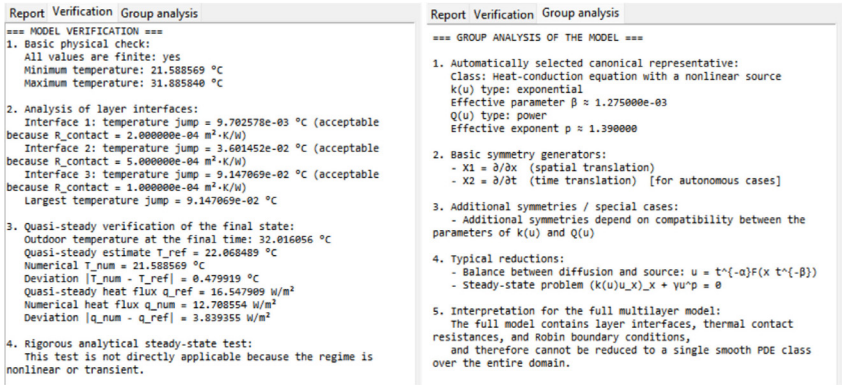


Figure 3.2. Fragment of the verification and group-analysis results in the software implementation

A separate program module is responsible for the scenario-based description of the thermal loading. The external temperature is specified as a function of time, and for selected scenarios a spatially localized internal heat source is also introduced. As a result, the numerical algorithm is not limited to calculating a single steady state but models the evolution of the temperature field under several loading regimes: constant, cyclic, pulsed, and localized-pulse. This formulation places the model within a scenario-analysis framework, which is essential for investigating thermal inertia, peak loads, and the ability of the structure to smooth short-duration thermal effects.

A major advantage of the developed software is that the numerical implementation is integrated with a verification block from the outset. This corresponds to the contemporary scientific standard according to which the correctness of a numerical scheme must be supported not only by general reasoning but also by direct tests against benchmark regimes. In this study, verification is organized into several successive levels: a basic physical check of the solution, analysis of interface consistency, quasi-steady

comparison of the final state with an analytical estimate based on the total thermal resistance, and an auxiliary transient semi-analytical benchmark.

The first level of verification consists in checking the physical correctness of the numerical solution. The finiteness of all temperature values and their range are monitored. This stage filters out possible indications of numerical instability, inconsistent parameters, or incorrectly specified boundary conditions. Although this is the most general check, it is important as an initial filter without which subsequent comparisons with benchmarks would lose their meaning.

The second verification level concerns the analysis of layer interfaces. If no thermal contact resistance is present at an interface, the temperature jump should be close to zero. If a contact resistance is specified, a nonzero temperature jump is physically admissible. In the present verification block, this condition is used as a qualitative consistency check according to whether R_c *is specified at the corresponding interface; a separate pointwise verification of the relation between the interface temperature jump and the local heat flux is not performed. This check therefore provides a qualitative assessment of whether the numerical solution is consistent with the physical meaning of the adopted interface model.*

The third verification level is a quasi-steady comparison of the final state with an analytical estimate based on the total thermal resistance of the structure. Effective thermal conductivities are calculated for each layer at its mean temperature, after which an equivalent linear structure is formed and the heat flux and inner-surface temperature are determined analytically. These quantities are then compared with the numerical result at the final time step. This approach is not an "exact" analytical solution of the complete nonlinear problem, but it makes it possible to verify whether the numerical model approaches a physically consistent quasi-steady regime when the thermal process has evolved sufficiently close to that state.

Finally, the fourth verification level uses an auxiliary transient benchmark for a homogeneous heat-conduction problem with prescribed boundary temperatures. The program separately implements an exact series solution and an auxiliary implicit numerical solver. Comparing them makes it possible to evaluate the absolute error, the L_∞ norm, and the root-mean-square deviation. This test provides the most rigorous validation of the spatiotemporal core of the numerical algorithm because

it checks not only an individual final state but a complete transient process [16, pp. 3175–3187; 20, pp. 1279–1296].

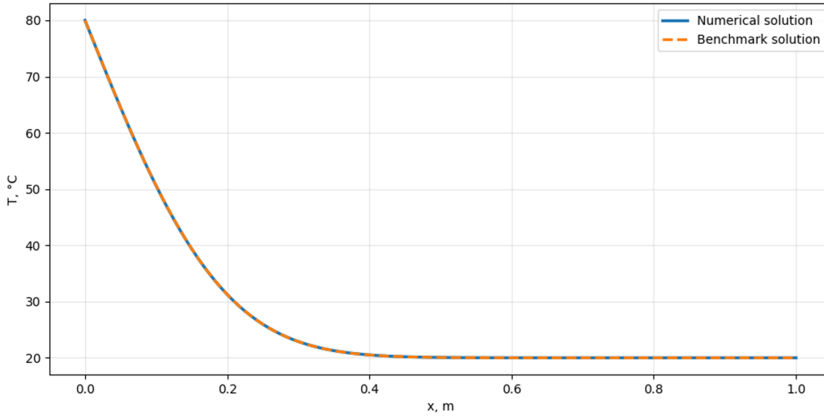


Figure 3.3. Numerical and benchmark temperature profiles

For the test transient problem, the numerical solution is compared with the benchmark temperature profile in Figure 3.3. The near-complete overlap of the curves indicates that the selected discretization scheme correctly reproduces the spatial temperature distribution and does not introduce systematic distortion into the profile shape. In this subsection, the figure serves as the primary confirmation of the reliability of the numerical core.

Figure 3.4 supplements the preceding comparison with a quantitative estimate of the difference between the numerical and benchmark solutions. It shows the distribution of absolute error along the spatial coordinate and makes it possible to determine whether the scheme contains localized regions of reduced accuracy, particularly near interfaces or boundary points. Together with Figure 3.3, this plot provides both visual and metric confirmation of the adequacy of the implementation. The convergence of the Picard iterations deserves separate attention (Figure 3.5). Because this procedure accounts for the nonlinearity of the temperature-dependent coefficients, monitoring its convergence is an essential component of algorithm verification. If the iterative process were unstable or excessively slow, the numerical scheme would not provide reliable consistency between

the model parameters and the current temperature state. The program therefore includes a test that tracks the maximum difference between two successive iterations [7, pp. 45–58].

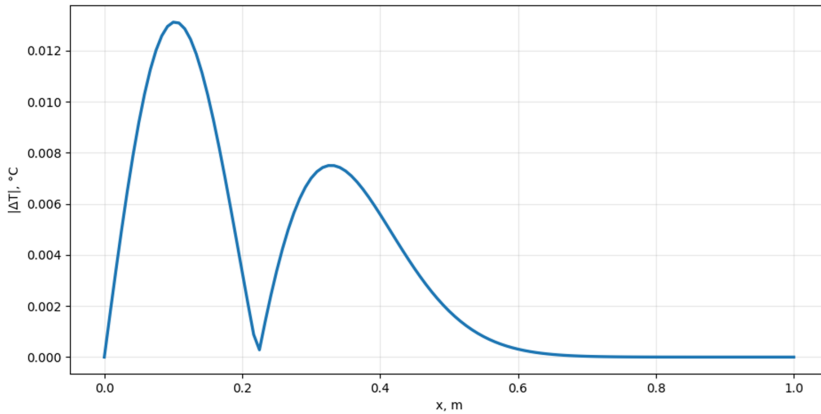


Figure 3.4. Absolute-error profile

The logarithmic plot clearly demonstrates the decay of the iterative error. The physical and mathematical meaning of Figure 3.5 is that the coefficients $\lambda(T)$ and $(\rho c)(T)$, which are updated at each step from the current temperature field, converge sufficiently rapidly to a self-consistent state. Thus, the nonlinear formulation does not compromise the numerical stability of the model and is incorporated correctly into the iterative scheme.

To complete the verification block, it is useful to move from a one-dimensional profile comparison to the spatiotemporal representation of the benchmark process. This makes it possible to verify that the numerical core reproduces not only an individual time instant but the entire evolution of the temperature field in the test problem. For this purpose, the program constructs a three-dimensional surface of the exact transient solution (Figure 3.6), which is used as a spatiotemporal reference for interpreting the results [16, pp. 3175–3187].

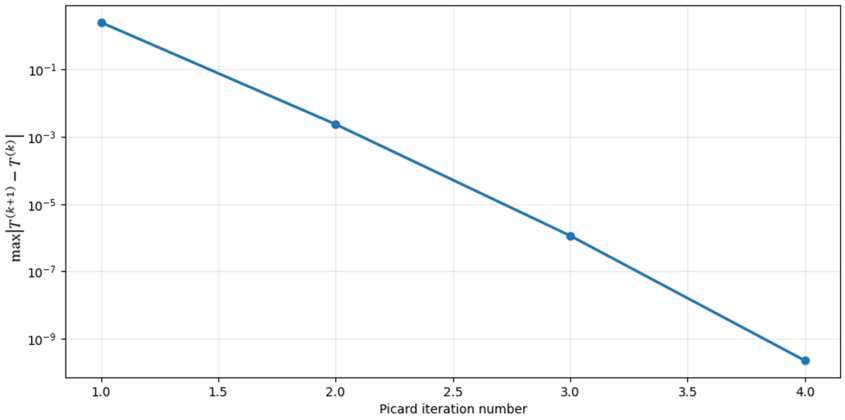


Figure 3.5. Convergence of Picard iterations

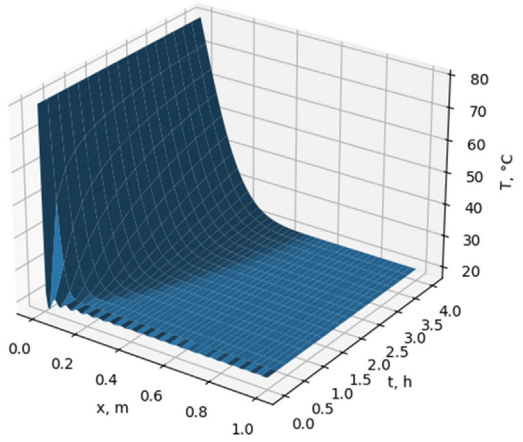


Figure 3.6. Benchmark spatiotemporal temperature distribution

The evolution of the benchmark temperature field is shown in the coordinates x and t. This figure completes the verification block because it presents the process as a whole, from the initial state through the subsequent heating of the medium. Its inclusion emphasizes that verification concerns

not only a single numerical profile but the entire spatiotemporal core of the model.

Overall, the numerical implementation combines several fundamentally important qualities. It is sufficiently flexible to describe multilayer media with temperature-dependent coefficients and contact resistances, sufficiently stable for transient scenarios involving severe thermal actions, and at the same time sufficiently transparent for multistage verification. This combination makes the software model not merely a computational tool but a full research instrument for analyzing nonlinear heat transfer in multilayer structures [8, pp. 1–8; 19, pp. 1–8].

3.4. Numerical Simulation Results and Physical Interpretation

Once the mathematical formulation has been constructed and the correctness of the numerical algorithm has been confirmed, the simulation results can be analyzed for physically meaningful thermal-loading scenarios. In this study, the scenario block includes several characteristic regimes: a steady external thermal background, a daily temperature cycle, a short-duration external thermal pulse, and a localized high-temperature pulse that may be accompanied by an internal volumetric heat source within part of the structural thickness. This range of regimes makes it possible to investigate not only limiting temperature states but also the time-dependent response of the structure to variable loading, its thermal inertia, and its ability to smooth peak effects.

The first integral indicator to be considered is the inner-surface temperature. It is most directly related to the thermal state of the internal space and is therefore a principal performance characteristic of the multilayer structure. If the system has high thermal inertia and sufficient total thermal resistance, then even under intense external forcing the inner-surface temperature rises more slowly, reaches a lower maximum, and crosses a critical threshold later. The time history of this quantity therefore concentrates the combined effects of all model mechanisms: internal diffusion, layer heat capacities, contact resistances, and external loading scenarios [4, pp. 1–10; 17, pp. 3373–3391].

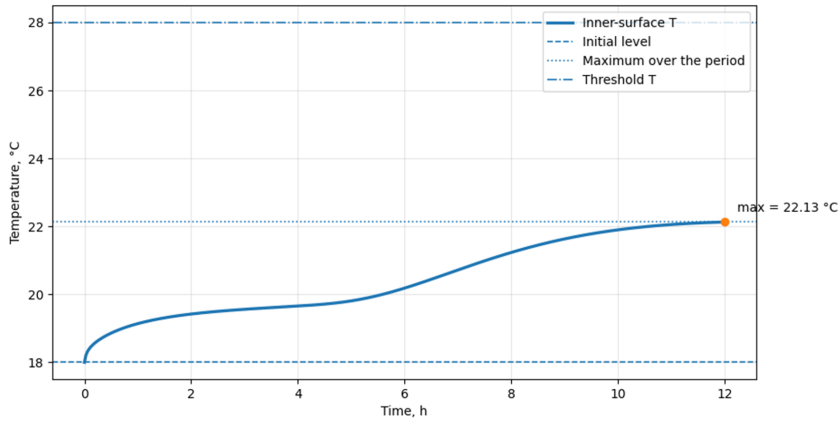


Figure 3.7. Evolution of the inner-surface temperature

Figure 3.7 shows the time evolution of the inner-surface temperature. This curve is a principal integral simulation result because it demonstrates how effectively the structure delays penetration of the thermal wave into the internal space. The maximum of the curve characterizes the peak thermal load, whereas the time at which the threshold is reached characterizes the delay of the thermal response, one of the principal performance criteria for a multilayer medium.

The inner-surface temperature, however, describes only the "kinematic" aspect of the process. A more complete energetic interpretation also requires analysis of the heat fluxes at the outer and inner boundaries. Comparing the incoming outer heat flux with the inner heat flux penetrating into the internal space makes it possible to assess the barrier role of the layers, the effect of contact resistances, and heat storage within the structure. In temporal terms, part of the energy passes through the system almost immediately, whereas another part is temporarily retained in the medium, thereby producing thermal inertia [1, p. 125500; 5, pp. 1001–1009].

The time variation of heat fluxes through the outer and inner surfaces of the structure is shown in Figure 3.8. The physical meaning of this figure lies in its representation of the system energy balance. For scenarios without internal volumetric heat generation, the difference between the outer and

inner heat fluxes characterizes the rate of heat accumulation within the structural layers. For the localized-pulse scenario with an internal heat source, the energy balance additionally includes the volumetric heat-generation term. Owing to thermal accumulation within the multilayer medium, the structure can smooth short-duration thermal pulses and reduce the instantaneous thermal load on the internal space.

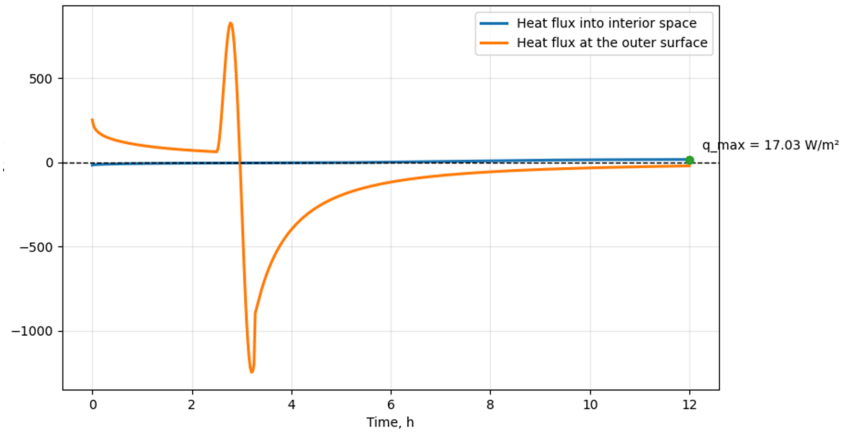


Figure 3.8. Evolution of heat fluxes at the outer and inner surfaces

The next step in the physical interpretation is to move from integral time-dependent characteristics to a spatial analysis of the temperature field (Figure 3.9). The program therefore stores through-thickness temperature profiles at several characteristic times. These profiles directly reveal how the thermal disturbance penetrates the system, which layers behave as relatively fast-conducting regions, and which act as inertial or barrier layers. If thermal contact resistances are present, characteristic temperature jumps may also appear at layer interfaces [9, pp. 1–8; 10, pp. 1–9].

A family of temperature profiles corresponding to different stages of the transient process is presented. The figure makes it possible to trace the gradual penetration of the thermal wave into the structure, assess the role of individual layers in temperature redistribution, and determine how the profile shape evolves with time. Both the effect of thermal inertia and the

influence of interlayer boundaries and contact resistances are particularly evident in this representation.

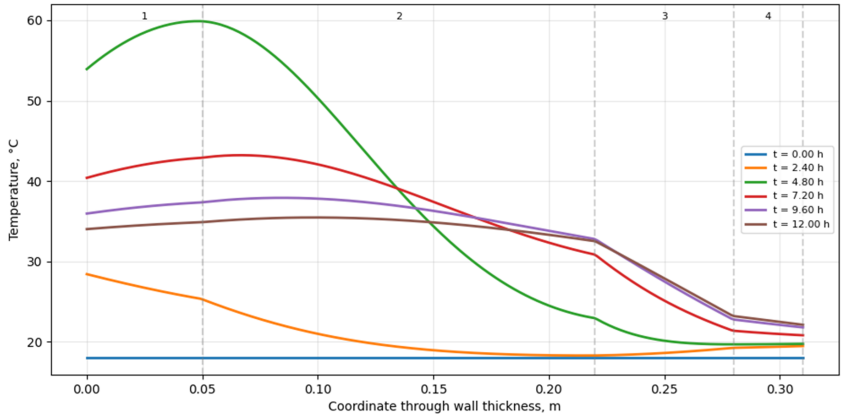


Figure 3.9. Temperature distribution through the structural thickness at different times

The most generalized representation of the results is a map of the temperature field in space-time coordinates. Unlike individual profiles, it presents the entire process as a single evolutionary picture in which the propagation rate of the thermal wave, time delays, localized regions of high gradient, and the depth of disturbance penetration into the structure are clearly visible. This form is especially useful for comparing different loading scenarios and material configurations because it provides an integral view of the dynamics of the complete system [8, pp. 1–8; 19, pp. 1–8]. Figure 3.10 presents the spatiotemporal map of the temperature field through the structural thickness. It is the most generalized figure in the results block because it simultaneously shows both the temporal development of the thermal process and the spatial structure of the temperature distribution. In this representation, the thermal-wave propagation rate, the inertial delay of the inner layers, and the overall barrier role of the multilayer system are most clearly apparent.

Comparison of the results for different scenarios leads to several principal conclusions. First, under a constant external thermal load the

structure gradually approaches a quasi-steady regime, in which the total thermal resistance plays the dominant role. Second, under daily or other cyclic effects, thermal inertia becomes decisive: layers with high heat capacity can both reduce the amplitude of temperature oscillations and shift the time of the inner-temperature maximum.

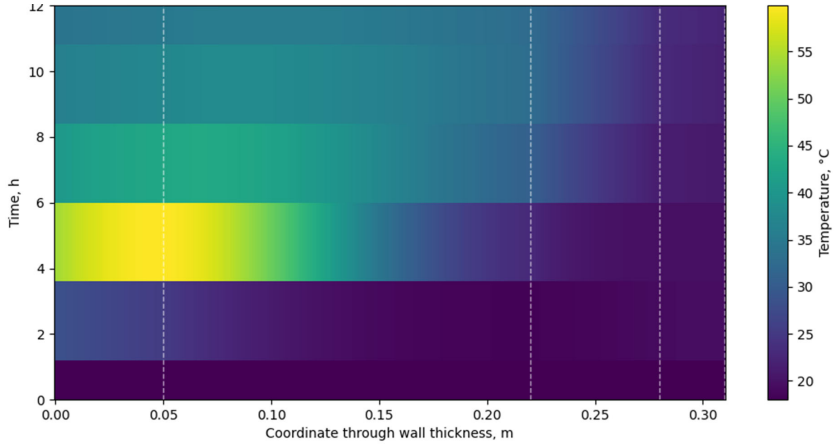


Figure 3.10. Temperature-field map within the wall

Third, under short-duration pulsed effects, structural performance is governed not only by total thermal resistance but primarily by the ability to localize, store, and spread the thermal pulse over time. Under these conditions, multilayer structure and temperature dependence of the properties become particularly important [4, pp. 1–10; 17, pp. 3373–3391].

An important final parameter is the integral thermal index, defined in the software by integrating the positive inward heat flux over time. Unlike peak temperature, this indicator represents the accumulated rather than instantaneous thermal action on the internal space. Together with the time required to reach a threshold temperature, it allows the structure to be evaluated not only in terms of a single maximum but also in terms of the cumulative thermal load over the entire calculation period. Such a multicriteria interpretation is particularly appropriate for transient thermal processes [17, pp. 3373–3391].

CONCLUSIONS

The results of the study show that analysis of the thermal state of multilayer protective structures requires a transition from simplified linear and steady-state schemes to consistent nonlinear transient models that simultaneously account for the heterogeneous layered structure of the medium, the temperature dependence of thermophysical properties, interlayer thermal contact resistances, and time-varying external thermal effects. This approach allows a multilayer structure to be treated not merely as a sequence of materials but as a dynamic medium with distributed barrier and inertial properties that is sensitive to the loading regime, layer arrangement, and character of interlayer interaction [7, pp. 45–58; 16, pp. 3175–3187].

It has been shown that classical heat-conduction models for homogeneous and multilayer media provide the necessary theoretical foundation for constructing an applied model, but their independent applicability to complex transient regimes is limited. The linear heat equation, steady-state formulations, and constant-coefficient models correctly describe simplified regimes but cannot fully reproduce thermal inertia, temporal lag of the temperature response, smoothing of short-duration pulses, or the influence of temperature-dependent material properties [5, pp. 1001–1009; 16, pp. 3175–3187].

It has been substantiated that inclusion of the temperature-dependent thermal conductivity λ(T) and effective volumetric heat capacity (ρc)(T) is essential becaus these quantities move the problem into the quasilinear nonlinear class and provide a physically adequate description of heat transfer in real materials. The canonical equation of the form $u_t = \left(K(u)u_x\right)_x + Q(u)$ *has been shown to be suitable as the theoretical core of the constructed model because it retains the principal features of nonlinear diffusion and allows the complete multilayer formulation to be interpreted in the broader context of mathematical physics [2, pp. 192–213; 11, pp. 347–358].*

It has been confirmed that interlayer thermal contact is not an auxiliary but a structurally important component of the model. Interface conditions with thermal contact resistance make it possible to move from an idealized formulation with continuous temperature to a physically meaningful description of a real layer boundary at which an additional thermal

barrier may form. Contact resistance produces local temperature drops at interfaces and contributes to the overall thermal resistance and transient thermal response of the multilayer system [5, pp. 1001–1009; 16, pp. 3175–3187].

It has been shown that the most appropriate approach for studying the complete nonlinear transient problem is a numerical implementation based on implicit time discretization and iterative updating of the temperature-dependent coefficients. The Backward Euler scheme combined with a Picard-type iterative procedure provides stable computation for multilayer media with sharp contrasts in layer properties and makes it possible to simulate both quasi-steady regimes and short-duration intense thermal pulses. The verification block, based on comparison of the numerical solution with analytical and semi-analytical benchmarks, confirmed the correctness of the spatiotemporal numerical core and provided a qualitative consistency check of the behavior at layer interfaces [7, pp. 45–58; 20, pp. 1279–1296].

The numerical simulation results have a clear physical meaning and allow the multilayer structure to be interpreted as a system with its own thermal inertia, time lag, and ability to redistribute thermal loading. The maximum inner-surface temperature, time to reach a critical threshold, heat fluxes at the outer and inner surfaces, and the integral thermal index should be regarded as key performance indicators of such a system. The numerical results illustrate the smoothing of pulsed thermal effects, limitation of peak inner-surface heating, and delayed thermal response in the investigated multilayer configuration; these effects are consistent with the thermal inertia of the layers and the presence of additional barrier regions [17, pp. 3373–3391; 13, pp. 44–81].

It has been confirmed that the Python implementation of the model provides a practical transition from the theoretical formulation to systematic numerical investigation. The developed software package makes it possible to define a multilayer configuration, specify material parameters and contact resistances, select a thermal-loading scenario, calculate temperature fields, perform verification, and obtain a group-theoretical interpretation of the model. In this form, the program serves not merely as a computational shell but as a complete scientific and methodological tool for investigating nonlinear heat transfer in multilayer structures.

Thus, the principal result of the study is the construction of a consistent nonlinear transient heat-transfer model for a multilayer medium that combines classical heat-conduction theory, temperature-dependent transport coefficients, interlayer thermal contact resistances, scenario-based thermal actions, numerical verification, and canonical mathematical interpretation within a unified methodological framework. This integration bridges the gap between classical linear schemes, the actual nonlinear properties of multilayer media, and applied numerical-analysis tools, thereby providing a basis for further development of physically grounded models of protective structures and heterogeneous heat-conducting systems [11, pp. 347–358; 14, pp. 1–6].

Further research should extend the model to more complex geometrical formulations, stronger nonlinearities in thermophysical properties, phase transformations, and more complex source terms, while further developing analytical-numerical approaches in which canonical symmetry analysis, benchmark solutions, and high-accuracy computational schemes operate as a unified research environment [1, p. 125500; 14, pp. 1–6].

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