APPROXIMATIVE CHARACTERISTICS OF CLASSES OF PERIODIC FUNCTIONS

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DOI: https://doi.org/10.30525/978-9934-26-715-4-58

Abstract. Approximation theory is one of the fundamental and classical fields of modern mathematical analysis. While in the 20th century it developed primarily to solve theoretical problems in analysis and computational mathematics, in the 21st century it has transformed into a foundational tool for artificial intelligence, Big Data processing, and computer modeling.

The study of periodic functions occupies an important place in this field. A breakthrough stage in the development of this direction was the introduction of a classification of periodic functions using the parameters (ψ,β) by the prominent Ukrainian mathematician O. I. Stepanets. The investigation of exact order approximative characteristics is an extremely relevant task of modern mathematics, as it creates a theoretical foundation for computational mathematics, coding theory, and information compression methods.

The purpose of this research is to establish exact order estimates of basic approximative characteristics for the classes of periodic functions $L^{\psi}_{\beta,p}$ in the metrics of Lebesgue spaces L_q under various relations between the parameters p and q. The main subject of the research includes the best approximations, approximations by Fourier sums, the best m-term trigonometric approximations, and the best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,p}$ in the space L_q for various relations between the parameters p and q.

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This paper utilizes a complex of methods of modern function theory combined with the methods developed in the works of O. I. Stepanets, V. N. Temlyakov, E. S. Belinsky, A. S. Romanyuk, and others.

We obtain the exact order estimates of the best approximations, approximations by Fourier sums and the best orthogonal trigonometric approximations of 2π periodic functions $F_{\psi}(x,\beta)$ in the space L_q by $1 < q < \infty$. And establish the exact order estimates of the best approximations, approximations by Fourier sums, the best m -term trigonometric approximations, the best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,p}$ in the space L_q for some relations between parameters p and q.

The obtained results and the methodology of their derivation can be used in the further study of problems in approximation theory, as well as practically in almost every field involving computers, computation, and data analysis.

INTRODUCTION

Approximation problems formulated on classes of functions are, in many cases, problems that seek to find or estimate the supremum of the approximation error for a given method on a fixed class of functions.

Among existing methods of approximation by trigonometric polynomials, the simplest and most natural method for approximating functions from a given class is the Fourier method, which consists in approximating functions from this class by a partial sum of their Fourier series. Regarding the problem of approximating certain classes of functions by Fourier sums, as well as by polynomials constructed on their basis, many deep and definitive results are known, which can be found in papers [7; 12].

Recently, the method of m -term trigonometric approximation has become increasingly widespread in approximation theory. The approximative properties of this method with respect to well-known classes of periodic functions, $W^r_{\beta,p}$, H^r_p and $B^r_{p,\theta}$, of both single and multiple variables, have been investigated in the works [5; 11].

In this context, it is important to note that for the aforementioned classes of functions, the advantages of m -term trigonometric approximations over classical approximation methods have been revealed in several important cases.

In connection with the best m-term trigonometric approximation, E. S. Belinsky [2; 12] introduced and considered the notion of the best orthogonal trigonometric approximation.

In 1983, O. I. Stepanets introduced a new classification of periodic functions of a single variable [3]. As a result, the classes $L^{\psi}_{\beta,p}$ were introduced, which, for fixed values of the parameters defining them, coincide with the Weyl–Nagy classes $W^{r}_{\beta,p}$. For these classes $L^{\psi}_{\beta,p}$, solutions to a wide range of problems in approximation theory, which were previously studied for the Weyl–Nagy classes, have currently been obtained.

Before proceeding to the presentation of the main results, we present necessary notations and we will give definitions of the approximate characteristic which will be investigated.

Let L_q be the space of $2\pi-$periodic functions f summable to a power q, $1 \le q < \infty$ (resp., essentially bounded for $q = \infty$), on the segment $[-\pi, \pi]$. The norm in this space is defined as follows:

$$\| f \|_{L_q} = \| f \|_q = = \begin{cases} \left(\frac{1}{2\pi} | \int\limits_{-\pi}^{\pi} f(x) |^q \, dx \right)^{\frac{1}{q}}, 1 \le q < \infty, \\ \operatorname{ess\,sup}\limits_{x \in [-\pi, \pi]}, \qquad q = \infty. \end{cases}$$

For a function $f \in L_1$, we consider its Fourier series

$$\sum_{k \in \mathbb{Z}} f(k) e^{ikx},$$

where

$$f(k) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

are the Fourier coefficients of the function $f \in L_1$. In what follows, we always assume that the function $f \in L_1$ satisfies the condition

$$\int\limits_{-\pi}^{\pi} f(x) dx = 0.$$

Further, let $\psi \neq 0$, be an arbitrary function of natural argument and let β be an arbitrary fixed real number. If a series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{f(k)}{\psi(|k|)} e^{i\left(kx+\beta\frac{\pi}{2}\operatorname{sign}k\right)}$$

is the Fourier series of a summable function, then, following Stepanets [9] we can introduce the (ψ,β)–derivative of the function f and denote it by f_β^ψ. By L_β^ψ we denote the set of functions f satisfying this condition. In what follows we assume that the function f belongs to the class $L_{\beta,p}^\psi$ if $f\in L_{\beta,p}^\psi$ and

$$f_\beta^\psi \in U_p = \left\{\varphi : \varphi\in L_p, \|\varphi\|_p \le 1\right\},\ \ 1\le p\le\infty.$$

If $\psi(|k|)=|k|^{-r}$, $r\rangle 0$, $k\in\mathbb{Z}\setminus\{0\}$, then the (ψ,β)– derivative of the function f coincides with its (r,β)–derivative (denoted by f_β^r) in the Weyl–Nagy sense.

Let ψ be a fixed function of a natural argument and $\beta\in\mathbb{R}$ be a given number such that the series

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \psi(|k|) e^{-i\frac{\pi}{2}\beta\operatorname{sign}k} e^{ikx}$$

is the Fourier series of some function $F_\psi(x,\beta)$ that is integrable on $[-\pi,\pi]$. Then every function $f\in L_{\beta,p}^\psi$ can be represented as a convolution

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x-t)F_\psi(t,\beta)\,dt, \tag{1}$$

where $\|\varphi\|_p\le 1$, $1\le p\le\infty$ (see, e.g., [9]).

We note that it is natural to call the functions $F_\psi(x,\beta)$ analogs of the Bernoulli kernels, since for $F_\psi(x,\beta)$, the function $F_\psi(x,\beta)$ is the well-known Bernoulli kernel.

Let B denote the set of functions $\psi(t)$, $t\in\mathbb{N}$, satisfying the following conditions:

1) ψ is positive and non-increasing;

2) there exists a constant $C>0$ such that

$$\frac{\psi(t)}{\psi(2t)}\le C,\qquad t\in\mathbb{N}.$$

Thus, the functions $\frac{1}{\tau^r}, r>0; \frac{\ln^\gamma(\tau+1)}{\tau^r}, \gamma\in\mathbb{R},\ r>0, \tau\in\mathbb{N},$ and some other functions belong to the set B.

For the quantities A and B, the notation $A \asymp B$ means that there exist positive constants C_1 and C_2 such that $C_1 A \le B \le C_2 A$. If $B \le C_2 A$ $\left(B \ge C_1 A\right)$, than we can write $B \ll A$ $\left(B \gg A\right)$. All C_i, $i=1,2,\ldots$, encountered in our paper may depend only on the parameters appearing in the definitions of the class and metric in which we determine the error of approximation.

We define the approximative characteristic that will be investigated in this paper.

Let T_m be the set of trigonometric polynomials t of the form

$$t(x)=\sum_{k=-m}^{m} c_k e^{ikx}.$$

For $f\in L_q$, $1\le q\le\infty$, let us set

$$E_m(f)_q=\inf_{t\in T_m}\| f(\cdot)-t(\cdot)\|_q .$$

This quantity is called the best approximation of the function f by the set of polynomials T_m in the space L_q. Monographs by Temlyakov V. N. provide investigations of the quantities $E_m(f)_q$ for some important functions.

If $L^{\psi}_{\beta,p}\subset L_q$, $1\le q\le\infty$, then let us set

$$E_m(L^{\psi}_{\beta,p})_q=\sup_{f\in L^{\psi}_{\beta,p}}\inf_{t\in T_m}\| f(\cdot)-t(\cdot)\|_q .$$

Further, by $\mathcal{E}_m(L^{\psi}_{\beta,p})_q$ we denote the quantity

$$\mathcal{E}_m(L^{\psi}_{\beta,p})_q=\sup_{f\in L^{\psi}_{\beta,p}}\| f(\cdot)-\sum_{k=-m}^{m} f(k)e^{ik\cdot}\|_q,$$

which is called the approximation of functional classes $L^{\psi}_{\beta,p}$ by Fourier sums in the space L_q.

For the quantities $E_m(L^{\psi}_{\beta,p})_q$ and $\mathcal{E}_m(L^{\psi}_{\beta,p})_q$, the following relation holds:

$$E_m(L^{\psi}_{\beta,p})_q\le\mathcal{E}_m(L^{\psi}_{\beta,p})_q.$$

In addition to the aforementioned approximative characteristics, we will also investigate the following quantities

$$e_m(f)_q = \inf_{\Theta_m} \inf_{T(\Theta_m,\cdot)} \| f(\cdot) - T(\Theta_m,\cdot) \|_q \ , \tag{2}$$

$$e_m^{\perp}(f)_q = \inf_{\Theta_m} \| f(\cdot) - S_{\Theta_m}(f,\cdot) \|_q \ , \tag{3}$$

where

$$T(\Theta_m, x) = \sum_{k=1}^{m} c_k e^{in_k x},$$

Θ_m is a set of m integers $n_1, ..., n_m$ and c_k are arbitrary complex numbers,

$$S_{\Theta_m}(f,x) = \sum_{k=1}^{m} f(n_k) e^{in_k x}.$$

The quantity (2) is called the best m -term trigonometric approximation, and (3) is called the best orthogonal trigonometric approximation of a function $f \in L_q$. If $L^{\psi}_{\beta,p} \subset L_q$, then let us set

$$e_m(L^{\psi}_{\beta,p})_q = \sup_{f \in L^{\psi}_{\beta,p}} e_m(f)_q, \tag{4}$$

$$e_m^{\perp}(L^{\psi}_{\beta,p})_q = \sup_{f \in L^{\psi}_{\beta,p}} e_m^{\perp}(f)_q. \tag{5}$$

It is obvious that the quantities (4) and (5) are related by

$$e_m(L^{\psi}_{\beta,p})_q \le e_m^{\perp}(L^{\psi}_{\beta,p})_q.$$

1. Best approximations and approximations by Fourier sums and the best orthogonal trigonometric approximations of 2π periodic functions $F_{\psi}(x, \beta)$ in the space L_q by $1 < q < \infty$.

In the notation adopted above, the following theorem holds.

Theorem 1. Let $1 < q < \infty$, $\psi \in B$, $\beta \in \mathbb{R}$ and let, in addition, there exist $\varepsilon > 0$ such that the sequence $\psi(t) t^{1-\frac{1}{q}+\varepsilon}$, $t \in \mathbb{N}$, does not increase. Then the following order estimate is true:

$$\mathcal{E}_m(F_{\psi})_q \asymp E_m(F_{\psi})_q \asymp \psi(m) m^{1-\frac{1}{q}}.$$

Proof. We prove an upper bound for the quantity $\mathcal{E}_m(F_{\psi})_q$. Let l and m be such that $2^l < m \le 2^{l+1}$. First, consider the case $1 < q \le 2$. Applying The Littlewood-Paley Theorem [6, p. 134], we obtain

$$E_m(F_\psi)_q \ll \| F_\psi - \sum_{s<l} \delta_s\left(F_\psi\right) \|_q = \| \sum_{s\geq l} \delta_s\left(F_\psi\right) \|_q \ll$$

$$\ll \| \left(\sum_{s\geq l} \left| \delta_s\left(F_\psi\right)\right|^2\right)^{\frac{1}{2}} \|_q = I_1.$$

Next, using the inequality $\left|a+b\right|^\alpha \leq \left|a\right|^\alpha + \left|b\right|^\alpha$ for $0 \leq \alpha \leq 1$, we can write

$$I_1^q \ll \int_\pi^\pi \sum_{s\geq l} \left| \delta_s\left(F_\psi\right)\right|^q dx \ll \sum_{s\geq l} \| \delta_s\left(F_\psi\right) \|_q^q,$$

that is,

$$I_1 \ll \left(\sum_{s\geq l} \| \delta_s\left(F_\psi\right) \|_q^q \right)^{\frac{1}{q}}. \tag{6}$$

To proceed further, we establish an estimate for the quantity

$$\| \delta_s\left(F_\psi\right) \|_q = \| \sum_{k\in\rho(s)} \psi\left(\left|k\right|\right) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \|_q .$$

First, we show that the following relation holds:

$$\| \sum_{k\in\rho(s)} \psi\left(\left|k\right|\right) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \|_q \ll \psi\left(2^s\right) \| \sum_{k\in\rho(s)} e^{ikx} \|_q,\ 1 < q < \infty.$$

To this end, for each $s \geq l$, we consider the sequence $\left\{\lambda_k\right\}$ defined by

$$\left\{\lambda_k\right\} = \left\{ \frac{\psi\left(\left|k\right|\right)}{\psi\left(2^s\right)} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k},\ 2^{s-1} \leq \left|k\right| < 2^s \right\}.$$

It is easy to see that the sequence $\left\{\lambda_k\right\}$ satisfies the conditions of the Marcinkiewicz theorem [6, p. 148]. Obviously, for this purpose, it is sufficient to verify that conditions 1 and 2 of this theorem hold for positive k such that $2^{s-1} \leq k < 2^s$.

Since $\psi \in B$ and $2^{s-1} \leq k < 2^s$, we have

$$1)\ \left|\lambda_k\right| = \left| \frac{\psi\left(k\right)}{\psi\left(2^s\right)} e^{-i\frac{\pi}{2}\beta} \right| = \frac{\psi\left(k\right)}{\psi\left(2^s\right)} \leq \frac{\psi\left(2^{s-1}\right)}{\psi\left(2^s\right)} \leq M,$$

$$2)ž \sum_{k=2^{s-1}}^{2^s-1} \left|\lambda_k - \lambda_{k+1}\right| =$$

$$= \sum_{k=2^{s-1}}^{2^s-1} \left| \frac{\psi(k)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta} - \frac{\psi(k+1)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta} \right| \le$$

$$\le \frac{1}{\psi(2^s)} \sum_{k=2^{s-1}}^{2^s-1} \left(\psi(k) - \psi(k+1)\right) = \frac{1}{\psi(2^s)} \left(\psi(2^{s-1}) - \psi(2^s)\right) \le$$

$$\le \frac{\psi(2^{s-1})}{\psi(2^s)} \le M.$$

Applying the multiplier Λ_s defined by the sequence $\{\lambda_k\}$ to the polynomial $\sum_{k\in\rho(s)} e^{ikx}$, we obtain

$$\Lambda_s \sum_{k\in\rho(s)} e^{ikx} = \sum_{k\in\rho(s)} \frac{\psi(|k|)}{\psi(2^s)} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} =$$

$$= \frac{1}{\psi(2^s)} \sum_{k\in\rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx}.$$

Thus, we can write

$$\| \Lambda_s \sum_{k\in\rho(s)} e^{ikx} \|_q = \frac{1}{\psi(2^s)} \| \sum_{k\in\rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \|_q .$$

However, by the Marcinkiewicz theorem, the following estimate holds:

$$\| \delta_s(F_\psi) \|_q = \| \sum_{k\in\rho(s)} \psi(|k|) e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx} \|_q \ll$$

$$\ll \psi(2^s) \| \sum_{k\in\rho(s)} e^{ikx} \|_q .$$

$$\| \Lambda_s \sum_{k\in\rho(s)} e^{ikx} \|_q \le C_1(q) M \| \sum_{k\in\rho(s)} e^{ikx} \|_q .$$

Thus, in conclusion, we obtain

Next, using the latter estimate as well as the well-known relation (see, e.g., [10, p. 25])

$$\| \sum_{k\in\rho(s)} e^{ikx} \|_q \asymp 2^{s\left(1-\frac{1}{q}\right)}, \quad 1<q<\infty, \tag{7}$$

we obtain

$$\| \delta_s\left(F_\psi\right)\|_q \ll \psi\left(2^s\right)2^{s\left(1-\frac{1}{q}\right)}. \tag{8}$$

Combining relations (6) and (8), we have

$$I_1 = \| \sum_{s\geq l} \delta_s\left(F_\psi\right)\|_q \ll \left(\sum_{s\geq l} \psi^q\left(2^s\right)2^{qs\left(1-\frac{1}{q}\right)}\right)^{\frac{1}{q}}.$$

Taking into account that $\psi(t)t^{1-\frac{1}{q}+\varepsilon}$ is non-increasing, we can write

$$E_m(F_\psi)_q \ll I_1 \ll \psi\left(2^l\right)2^{l\left(1-\frac{1}{q}+\varepsilon\right)}\left(\sum_{s\geq l} 2^{-s\varepsilon q}\right)^{\frac{1}{q}} \ll$$

$$\ll \psi\left(2^l\right)2^{l\left(1-\frac{1}{q}\right)} \asymp \psi(m)m^{1-\frac{1}{q}}.$$

Now, let us consider the case $2<q<\infty$. Applying the Littlewood–Paley theorem and Minkowski's inequality, we obtain

$$E_m(F_\psi)_q \ll \| F_\psi - \sum_{s<l}\delta_s\left(F_\psi\right)\|_q = \| \sum_{s\geq l}\delta_s\left(F_\psi\right)\|_q \ll$$

$$\ll \| \left(\sum_{s\geq l} \delta_s\left(F_\psi\right)|^2\right)^{\frac{1}{2}} \|_q = \left(\| \sum_{s\geq l} \delta_s\left(F_\psi\right)|^2 \|_{\frac{q}{2}}\right)^{\frac{1}{2}} \ll$$

$$\ll \left(\sum_{s\geq l} \| \delta_s\left(F_\psi\right)|^2 \|_{\frac{q}{2}}\right)^{\frac{1}{2}} = \left(\sum_{s\geq l} | \delta_s\left(F_\psi\right)\|_q^2\right)^{\frac{1}{2}}.$$

Using (8) and repeating the arguments carried out for the case $1<q\leq 2$, we obtain the desired estimate:

$$E_m(F_\psi)_q \ll \psi(m)m^{1-\frac{1}{q}}, \ 2<q<\infty.$$

Thus, the upper bounds in Theorem 1 are established.

Turning to the lower bounds, we note that it is sufficient to obtain them for

the quantity $E_m(F_\psi)_q$. Let $t^* \in T_m$ be the polynomial of best approximation of the function $F_\psi(x,\beta)$ in the space L_q, $1 < q < \infty$, $2^{l-1} \le m \le 2^l$, $l \in \mathbb{N}$, that is,

$$E_m(F_\psi)_q = \inf_{t \in T_m} \| F_\psi - t \|_q = \| F_\psi - t^* \|_q,$$

and

$$F_2(x,\beta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-2} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} e^{ikx}.$$

Let us consider the quantity

$$J = \left(F_\psi - t^*, F_2 - S_{2^l}(F_2)\right) = \left(F_\psi, F_2 - S_{2^l}(F_2)\right) -$$
$$-\left(t^*,\ F_2 - S_{2^l}(F_2)\right) = \left(F_\psi,\ F_2 - S_{2^l}(F_2)\right).$$

On the one hand, by Hölder's inequality, we have

$$J \le \| F_\psi - t^* \|_q \| F_2 - S_{2^l}(F_2) \|_{q'} =$$
$$= E_{2^l}(F_\psi)_q \| F_2 - S_{2^l}(F_2) \|_{q'},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

And since

$$\| F_2 - S_{2^l}(F_2) \|_{q'} \ll 2^{-l\left(2-\frac{1}{q}\right)},$$

we have

$$J \ll E_m(F_\psi)_q 2^{-l\left(2-\frac{1}{q}\right)}. \tag{9}$$

On the other hand, for J, we can write

$$J \gg \sum_{k \in \mathbb{Z} \setminus \{0\}} \psi\left(|k|\right) |k|^{-2} \gg \sum_{s \ge l} \sum_{k \in \rho^+(s)} \psi(k) k^{-2} =$$
$$= \sum_{s \ge l} \sum_{k=2^{s-1}}^{2^s} \psi(k) k^{-2} \gg \sum_{s \ge l} \psi\left(2^s\right) 2^{-s} \gg \psi\left(2^l\right) 2^{-l}. \tag{10}$$

where $\rho^+(s) = \{k : 2^{s-1} \le k < 2^s\}$.

Taking into account relations (9) and (10), we obtain

$$\psi\left(2^l\right) 2^{-l} \ll J \ll E_m(F_\psi)_q 2^{-l\left(2-\frac{1}{q}\right)},$$

that is,

$$E_m(F_\psi)_q \gg \psi\left(2^l\right)2^{l\left(1-\frac{1}{q}\right)} \asymp \psi(m)m^{1-\frac{1}{q}}.$$

The theorem is proved.

Remark 1. In the case $\psi(|k|) = |k|^{-r},\ r\rangle 1-\frac{1}{q},\ 1 \le q \le \infty,$ the corresponding result was obtained by V. M. Temlyakov.

In the next assertion, we establish the order of the quantity $e_m^{\perp}(F_\psi)_q$.

Theorem 2. Let $1 < q < \infty,\ \psi \in B,\ \beta \in \mathbb{R}$ and let, in addition, there exist $\varepsilon > 0$ such that the sequence $\psi(t)t^{1-\frac{1}{q}+\varepsilon}$, $t \in \mathbb{N}$, does not increase. Then the following order estimate is true:

$$e_m^{\perp}(F_\psi)_q \asymp \psi(m)m^{1-\frac{1}{q}}.$$

Proof. The upper bound follows from the previous theorem and relation

$$e_{2m+1}^{\perp}(F_\psi)_q \le \mathcal{E}_m(F_\psi)_q.$$

Let us proceed to establishing the lower bound. Let Θ_m be an arbitrary set of m integers $n_1,...,n_m$. Using relation

$$\| f \|_q = \sup_{\|g\|_{q'} \le 1} \left| \int_{-\pi}^{\pi} f(x)\overline{g}(x)dx \right|, \tag{11}$$

where $\frac{1}{q}+\frac{1}{q'}=1$ and $\overline{g}$ is the complex conjugate of the function g, $f \in L_q,\ 1 \le q \le \infty$.

we can write

$$\| F_\psi - S_{\Theta_m}(F_\psi) \|_q =$$

$$= \sup_{\|g\|_{q'} \le 1} \left| \int_{-\pi}^{\pi} \left(F_\psi(x) - S_{\Theta_m}(F_\psi, x)\right)\overline{g}(x)dx \right|. \tag{12}$$

For a given m, we choose $l \in \mathbb{N}$ from the condition $2m < 2^l \le 4m$ and set

$$g(x) = C_2 2^{-\frac{l}{q}} \sum_{k \in \rho^+(l)} e^{ikx},$$

where $\rho^+(l)=\{k\in\mathbb{Z}: 2^{l-1}\le k<2^l\}$. It is easy to see that $\|g\|_{q'}\le 1$, $q'\in(1,\infty)$, for a certain choice of the constant $C_2>0$.

Indeed, using relation (7), we will have

$$\|g\|_{q'}=C_2 2^{-\frac{l}{q}}\|\sum_{k\in\rho^+(l)}e^{ikx}\|_{q'}\asymp 2^{-\frac{l}{q}}2^{l\left(1-\frac{1}{q'}\right)}=1,$$

that is, the function g, for some choice of the constant C_2, satisfies the inequality

$$\|g\|_{q'}\le 1.$$

Now, substituting the function g into (12), we obtain

$$\|F_\psi-S_{\Theta_m}(F_\psi)\|_q\ge\left|\int_{-\pi}^{\pi}\left(F_\psi(x)-S_{\Theta_m}(F_\psi,x)\right)\overline{g}(x)dx\right|=$$

$$=\left|\int_{-\pi}^{\pi}F_\psi(x)\overline{g}(x)dx-\int_{-\pi}^{\pi}S_{\Theta_m}(F_\psi,x)\overline{g}(x)dx\right|=$$

$$=\left|\left(\sum_{k\in\rho^+(l)}C_2\psi(k)2^{-\frac{l}{q}}e^{-i\frac{\pi}{2}\beta}\right)-\left(\sum_{k\in\Theta_m\cap\rho^+(l)}C_2\psi(k)2^{-\frac{l}{q}}e^{-i\frac{\pi}{2}\beta}\right)\right|\gg$$

$$\gg 2^{-\frac{l}{q}}\sum_{k\in\rho^+(l)\setminus\Theta_m}\psi(k)\gg 2^{-\frac{l}{q}}\psi\left(2^l\right)2^l=$$

$$=2^{l\left(1-\frac{1}{q}\right)}\psi\left(2^l\right)\asymp\psi(m)m^{1-\frac{1}{q}}.$$

The lower bound, and with it the theorem, is proved.

Remark 2. In the case $\psi(|k|)=|k|^{-r}$, $r\rangle 1-\frac{1}{q}$, the order of the quantity $e_m^{\perp}(F_\psi)_q$, $1<q<\infty$, was established by A. S. Romanyuk.

2. The best orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,1}$ in the space L_q, $1<q<\infty$

Further, exact order estimates for the quantities $e_m^{\perp}(L^{\psi}_{\beta,1})_q$, for $1<q<\infty$ are established by applying Theorem 2. At the same time, we note that in the case $p=1$, additional conditions will be imposed on the functions ψ in addition to their membership in the set B, which is due to the use of the corresponding auxiliary assertions.

Theorem 3. Let $1 < q < \infty$, $\psi \in B$, $\beta \in \mathbb{R}$, one of the conditions

$$\Delta^2\left(\frac{1}{\psi(t)}\right) \geq 0,\ t \in \mathbb{N}, \tag{13}$$

or

$$\Delta^2\left(\frac{1}{\psi(t)}\right) \leq 0,\ t \in \mathbb{N}, \tag{14}$$

where

$$\Delta^2\left(\frac{1}{\psi(t)}\right) = \frac{1}{\psi(t)} - \frac{2}{\psi(t+1)} + \frac{1}{\psi(t+2)},$$

holds, and, moreover, there exists $\varepsilon > 0$ such that the sequence $\psi(t)t^{1-\frac{1}{q}+\varepsilon}$, $t \in \mathbb{N}$, is non-increasing. Then the following relation holds:

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q \asymp \psi(m)m^{1-\frac{1}{q}}.$$

Proof. We carry out the upper bound in a more general case. For this purpose, we need an auxiliary assertion.

Lemma 1. Let $1 \leq q < \infty$, $\psi \in B$, $\beta \in \mathbb{R}$. Then the following relation holds:

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q \leq e_m^{\perp}(F_{\psi})_q.$$

Proof. According to the definition of the class $L_{\beta,1}^{\psi}$, we can write

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q = \sup_{f \in L_{\beta,1}^{\psi}} \inf_{S_{\Theta_m}(f)} \| f - S_{\Theta_m}(f) \|_q =$$

$$= \sup_{\|\varphi\|_1 \leq 1} \inf_{S_{\Theta_m}(\varphi * F_{\psi})} \| \varphi * F_{\psi} - S_{\Theta_m}(\varphi * F_{\psi}) \|_q =$$

$$= \sup_{\|\varphi\|_1 \leq 1} \inf_{S_{\Theta_m}(F_{\psi})} \| \varphi * \left(F_{\psi} - S_{\Theta_m}(F_{\psi})\right) \|_q^q.$$

Taking into account the property of the convolution, we will have

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q \leq \sup_{\|\varphi\|_1 \leq 1} \inf_{S_{\Theta_m}(F_{\psi})} \| \varphi \|_1 \times \| F_{\psi} - S_{\Theta_m}(F_{\psi}) \|_q \leq$$

$$\leq \inf_{S_{\Theta_m}(F_{\psi})} \| F_{\psi} - S_{\Theta_m}(F_{\psi}) \|_q = e_m^{\perp}(F_{\psi})_q.$$

The lemma is proved.

Thus, the upper bound in Theorem[3] is obtained by using Lemma1 and the estimate of the quantity $e_m^{\perp}(F_{\psi})_q$ established in Theorem 2:

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q \le e_m^{\perp}(F_{\psi})_q \asymp \psi(m)m^{1-\frac{1}{q}}.$$

Let us proceed to establishing the lower bound. For a given m, we choose l from the condition $2m < 2^l \le 4m$ and consider the function

$$f_l(x) = C_3\psi\left(2^l\right)\left(V_{2^{l+1}}(x) - V_{2^l}(x)\right),\ C_3 > 0,$$

where $V_m(x)$ is the de la Vallée Poussin kernel of the form

$$V_m(x) = 1 + 2\sum_{k=1}^{m}\cos kx + 2\sum_{k=m+1}^{2m-1}\left(\frac{2m-k}{m}\right)\cos kx.$$

We show that $f_l \in L_{\beta,1}^{\psi}$ for a certain choice of the constant $C_3 > 0$.

For this purpose, it is sufficient to verify that $\| (f_l)_{\beta}^{\psi} \|_1 \le C_4,\ C_4 > 0$.

It is obvious that

$$\| (f_l)_{\beta}^{\psi} \| = C_4\psi\left(2^l\right)\| V_{2^{l+1}} - V_{2^l} \|_1 \ll \frac{\psi\left(2^l\right)}{\psi\left(2^{l+2}\right)}\| V_{2^{l+1}} - V_{2^l} \|.$$

Since $\| V_{2^{l+1}} - V_{2^l} \|_q \asymp 2^{l\left(1-\frac{1}{q}\right)}$, $1 \le q \le \infty$ (15) holds, we have $\| (f_l)_{\beta}^{\psi} \|_1 \ll 1$ and $f_l \in L_{\beta,1}^{\psi}$ for a certain choice of the constant $C_3 > 0$.

Next, let us consider the polynomial

$$g_1(x) = C_5 m^{-\frac{1}{q}}\left(V_{2^{l+1}}(x) - V_{2^l}(x)\right),\ C_5 > 0.$$

It follows from (15) that for a certain choice of the constant C_5, the polynomial g_1 satisfies the inequality $\| g_1 \|_{q'} \le 1$. Using (11), we obtain

$$\| f_l - S_{\Theta_m}(f_l) \|_q \ge \left| \int_{-\pi}^{\pi}\left(f_l(x) - S_{\Theta_m}(f_l, x)\right)g_1(x)\,dx \right| =$$

$$= \left| \int_{-\pi}^{\pi} f_l(x)g_1(x)\,dx - \int_{-\pi}^{\pi} S_{\Theta_m}(f_l, x)g_1(x)\,dx \right| \gg$$

$$\gg m^{-\frac{1}{q}}\psi\left(2^l\right)\left(\| V_{2^{l+1}} - V_{2^l} \|_2^2 - m\right) \gg m^{-\frac{1}{q}}\psi\left(2^l\right)\left(2^{l+1} - m\right) \ge$$

$$\geq m^{-\frac{1}{q}}\psi\left(2^l\right)2^l \asymp \psi\left(2^l\right)2^{l\left(1-\frac{1}{q}\right)}.$$

Taking into account the choice of l, from the latter estimate we have

$$e_m^{\perp}(L_{\beta,1}^{\psi})_q \gg \psi(m)m^{1-\frac{1}{q}}.$$

The lower bound, and with it Theorem 3, is proved.

Remark 3. In the case $\psi\left(|k|\right)=|k|^{-r}$, $r>1-\frac{1}{q}$, corresponding result was previously obtained by A. S. Romanyuk.

3. The best m -term trigonometric approximations and the best orthogonal trigonometric approximations of functions from the classes $L_{\hat{a},p}^{\psi}$, $1<p\leq\infty$, in the uniform metric.

Further, we establish exact order estimates for the best m -term trigonometric approximations of functions from the classes $L_{\beta,p}^{\psi}$, $1<p\leq\infty$, in the uniform metric. In addition, by applying Theorem 3, we find the exact orders of the best orthogonal trigonometric approximations of functional classes $L_{\beta,p}^{\psi}$, $1<p<\infty$, in the L_{∞} metric. In the next assertion, we establish an exact order estimate for the quantity $e_m(L_{\beta,p}^{\psi})_{\infty}$, $1<p\leq\infty$.

Theorem 4. Let $1<p\leq\infty$, $\psi\in B$, $\beta\in\mathbb{R}$ and, moreover, there exists $\varepsilon>0$ such that the sequence $\psi(t)t^{a+\varepsilon}$, $t\in\mathbb{N}$, $a=max\left\{\frac{1}{p},\frac{1}{2}\right\}$, is non-increasing. Then the following order estimate holds:

$$e_m(L_{\beta,p}^{\psi})_{\infty} \asymp \psi(m)m^{(\frac{1}{p}-\frac{1}{2})_+},$$

where $b_+=\max\{b;0\}$.

Proof. We obtain the upper bound. First, consider the case $1<p\leq 2$. We choose $l\in\mathbb{N}$ from the inequality $2^{l-1}\leq m<2^l$ and for $j\in\mathbb{N}$ let us set

$$k_j=\begin{cases}0, & j\leq l,\\ \left[\dfrac{2^{\frac{j}{p}}\psi\left(2^j\right)}{2^{\frac{l}{p}}\psi\left(2^l\right)}2^l\right]+1, & l<j\leq\gamma l,\\ 0, & j>\gamma l,\end{cases} \qquad (16)$$

where $[d]$ is the integer part of the number d and $\gamma > 1$ is a certain number that will be defined below.

Then, taking into account the fact that $\psi(t)t^{\frac{1}{p}+\varepsilon}$ is non-increasing, we have

$$\sum_{j=1}^{\infty} k_j \ll \frac{2^l}{2^{\frac{l}{p}}\psi\left(2^l\right)} \sum_{l<j\le\gamma l} 2^{\frac{j}{p}}\psi\left(2^j\right) + \gamma l =$$

$$= \frac{2^l}{2^{\frac{l}{p}}\psi\left(2^l\right)} \sum_{l<j\le\gamma l} 2^{j\left(\frac{1}{p}+\varepsilon\right)}\psi\left(2^j\right)2^{-j\varepsilon} + \gamma l \ll$$

$$\ll \frac{2^l}{2^{\frac{l}{p}}\psi\left(2^l\right)} 2^{l\left(\frac{1}{p}+\varepsilon\right)}\psi\left(2^l\right) \sum_{l<j\le\gamma l} 2^{-j\varepsilon} + \gamma l \ll$$

$$\ll \frac{2^l}{2^{\frac{l}{p}}\psi\left(2^l\right)} 2^{l\left(\frac{1}{p}+\varepsilon\right)}\psi\left(2^l\right)2^{-l\varepsilon} + \gamma l \ll 2^l.$$

Thus, for $f \in L^{\psi}_{\beta,p}$, we can write

$$e_m(f)_\infty \le \sum_{l<j\le\gamma l} e_{k_j}(\delta_j(f))_\infty + \sum_{j>\gamma l} \| \delta_j(f) \|_\infty = I_2 + I_3.$$

First, we estimate the quantity I_2. To carry out the subsequent arguments, we use an estimate from [4] (Corollary 5.1), which, according to our notation, has the form

$$e_{k_j}(\delta_j(f))_\infty \ll (\frac{2^j}{k_j})^{\frac{1}{2}} \log\frac{2^j}{k_j} \| \delta_j(f) \|_2, \qquad 17 \qquad (17)$$

(here and below, $\log := \log_2$).

Next, using (17) and the well-known assertion

$$\| t \|_p \ll n^{\frac{1}{q}-\frac{1}{p}} \| t \|_q, \ 1 \le q \le p \le \infty \qquad (18)$$

we can write

$$I_2 \ll \sum_{l<j\le\gamma l} (\frac{2^j}{k_j})^{\frac{1}{2}} \log\frac{2^j}{k_j} \| \delta_j(f) \|_2 \ll$$

$$\ll \sum_{l<j\le\gamma l} \left(\frac{2^j}{k_j}\right)^{\frac{1}{2}} \log\frac{2^j}{k_j} 2^{j\left(\frac{1}{p}-\frac{1}{2}\right)} \| \delta_j(f) \|_p . \tag{19}$$

In order to continue the estimate, we show that the following relation holds:

$$\| \delta_j(f) \|_p \ll \psi\left(2^j\right) \| \delta_j\left(f_\beta^\psi\right) \|_p . \tag{20}$$

To this end, we consider $\delta_j\left(f_\beta^\psi\right)$ and the operator Λ, defined by the sequence

$$\{\lambda_k\} = \left\{ \frac{\psi(|k|)}{\psi\left(2^j\right)} e^{-i\frac{\pi}{2}\beta \operatorname{sign} k} \right\},\ 2^{j-1} \le |k| < 2^j,$$

which acts as a multiplier.

Therefore, applying the operator Λ to $\delta_j\left(f_\beta^\psi\right)$, we obtain

$$\Lambda\delta_j\left(f_\beta^\psi\right) = \Lambda \sum_{k\in\rho(j)} \frac{e^{i\frac{\pi}{2}\beta \operatorname{sign} k}}{\psi(|k|)} f(k) e^{ikx} =$$

$$= \sum_{k\in\rho(j)} \frac{1}{\psi\left(2^j\right)} f(k) e^{ikx} = \frac{1}{\psi\left(2^j\right)} \delta_j(f). \tag{21}$$

We note that in the proof of Theorem 2, it was shown that the sequence $\{\lambda_k\}$ satisfies the conditions of the Marcinkiewicz theorem.

Thus, according to the Marcinkiewicz theorem, the following inequality holds:

$$\| \ddot{E}\delta_j\left(f_\beta^\psi\right) \|_p \le C_6(p) \| \delta_j\left(f_\beta^\psi\right) \|_p . \tag{22}$$

On the other hand, by virtue of relation (21),

$$\| \ddot{E}\delta_j\left(f_\beta^\psi\right) \|_p = \frac{1}{\psi\left(2^j\right)} \| \delta_j(f) \|_p \tag{23}$$

Comparing (22) and (23), we arrive at the estimate (20).

Further, since relation (20) holds, taking into account (16) and the fact that $\psi \in B$ and $\psi(t) t^{\frac{1}{p}+\varepsilon}$ is non-increasing, we continue the estimate (19):

$$I_2 \ll \sum_{l<j\le\gamma l}(2^j 2^{-\frac{j}{p}}\psi^{-1}\left(2^j\right)2^{-l}2^{\frac{l}{p}}\psi\left(2^l\right))^{\frac{1}{2}}\times$$

$$\times\left(\log 2^j-\log\left(2^{\frac{j}{p}}\psi\left(2^j\right)2^{-\frac{l}{p}}\psi^{-1}\left(2^l\right)2^l\right)\right)2^{j\left(\frac{1}{p}-\frac{1}{2}\right)}\psi\left(2^j\right)\ll$$

$$\ll 2^{-\frac{l}{2}}2^{\frac{l}{2p}}\psi^{\frac{1}{2}}\left(2^l\right)\sum_{l<j\le\gamma l}\psi^{\frac{1}{2}}\left(2^j\right)2^{\frac{j}{2p}}\left(j-\frac{j}{p}+\frac{l}{p}-l\right)\ll$$

$$\ll 2^{-\frac{l}{2}}2^{\frac{l}{2p}}\psi^{\frac{1}{2}}\left(2^l\right)\sum_{l<j\le\gamma l}\psi^{\frac{1}{2}}\left(2^j\right)2^{\frac{j}{2}\left(\frac{1}{p}+\varepsilon\right)}2^{-\frac{j\varepsilon}{2}}\left(j-l\right)\ll$$

$$\ll 2^{-\frac{l}{2}}2^{\frac{l}{2p}}\psi^{\frac{1}{2}}\left(2^l\right)\psi^{\frac{1}{2}}\left(2^l\right)2^{\frac{l}{2}\left(\frac{1}{p}+\varepsilon\right)}\sum_{l<j\le\gamma l}2^{-\frac{j\varepsilon}{2}}\left(j-l\right)\ll\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}. \quad (24)$$

Let us proceed to estimating the quantity I_3. Successively using the inequality of different metrics (18), the estimate (20), and also taking into account that $\psi(t)t^{\frac{1}{p}+\varepsilon}$ is non-increasing, we have

$$I_3=\sum_{j>\gamma l}\|\delta_j(f)\|_\infty\ll\sum_{j>\gamma l}2^{\frac{j}{p}}\|\delta_j(f)\|_p\ll$$

$$\ll\sum_{j>\gamma l}2^{\frac{j}{p}}\psi\left(2^j\right)\|\delta_j\left(f_\beta^\psi\right)\|_p\ll\sum_{j>\gamma l}2^{\frac{j}{p}}\psi\left(2^j\right)\ll$$

$$\ll\psi\left(2^{\gamma l}\right)2^{\gamma l\left(\frac{1}{p}+\varepsilon\right)}\sum_{j>\gamma l}2^{-j\varepsilon}\ll\psi\left(2^{\gamma l}\right)2^{\gamma l\frac{1}{p}}. \quad (25)$$

Comparing (24), and (25), we obtain

$$e_m(f)_\infty\ll\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}+\psi\left(2^{\gamma l}\right)2^{\gamma l\frac{1}{p}}=$$

$$=\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\left(1+\psi\left(2^{\gamma l}\right)2^{\gamma l\frac{1}{p}}\left(\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\right)^{-1}\right)=$$

$$=\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\left(1+\psi\left(2^{\gamma l}\right)2^{\gamma l\left(\frac{1}{p}+\varepsilon\right)}2^{-\gamma l\varepsilon}\left(\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\right)^{-1}\right). \quad (26)$$

Since $\psi(t)t^{\frac{1}{p}+\varepsilon}$ is non-increasing, from (26) we arrive at the estimate

$$e_m(f)_\infty \ll \psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\left(1+\psi\left(2^l\right)2^{l\left(\frac{1}{p}+\varepsilon\right)}2^{-\gamma l\varepsilon}\left(\psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\right)^{-1}\right)\ll$$

$$\ll \psi\left(2^l\right)2^{l\left(\frac{1}{p}-\frac{1}{2}\right)}\left(1+2^{l\varepsilon(1-\gamma)}2^{\frac{l}{2}}\right).$$

Thus, setting $\gamma=\left(\frac{1}{2\varepsilon}+1\right)$, and taking into account the choice of l, from the latter estimate we obtain

$$e_m(f)_\infty \ll \psi(m)m^{\frac{1}{p}-\frac{1}{2}}. \qquad (27)$$

Now, let us consider the case $2<p<\infty$.. In this case, the desired estimate follows from the embedding $L^{\psi}_{\beta,p}\subseteq L^{\psi}_{\beta,2}$ and the estimate (27) for $p=2$.

The upper bound is proved.

To establish the lower bound in the case $1<p<\infty$, it is sufficient to use the result obtained by A. S. Fedorenko:

$$e_m(L^{\psi}_{\beta,p})_q \gg \psi(m)m^{(\frac{1}{p}-\frac{1}{2})_+},\ 1<q<\infty, \quad 28 \qquad (28)$$

and the relation $e_m(L^{\psi}_{\beta,p})_\infty \geq e_m(L^{\psi}_{\beta,p})_q$.

Let us consider the case $p=\infty$. According to Parseval's identity, we have

$$e_m(L^{\psi}_{\beta,\infty})_\infty \geq e_m(L^{\psi}_{\beta,\infty})_2 =$$

$$= \sup_{\|f\|_\infty\leq 1}\inf_{\{\Theta_m\}}\Big(\sum_{k\in\mathbb{Z}\setminus\Theta_m}|f(k)|^2\,\psi^2(m)\Big)^{\frac{1}{2}}.$$

Using the assertion proved in [3, p. 33] and taking the supremum over $g\in L_2$, we can write

$$e_m(L^{\psi}_{\beta,\infty})_\infty \gg \sup_{\|g\|_2\leq 1}\inf_{\{\Theta_m\}}\Big(\sum_{k\in\mathbb{Z}\setminus\Theta_m}|g(k)|^2\,\psi^2(m)\Big)^{\frac{1}{2}} = e_m(L^{\psi}_{\beta,2})_2.$$

Using the result (28), we arrive at the desired lower bound for the quantity $e_m(L^{\psi}_{\beta,\infty})_{\infty}$.

Theorem 4 is proved.

Remark 4. In the case $\psi(|k|) = |k|^{-r}$, $r \rangle \frac{1}{p}$, the order of the quantity $e_m(L^{\psi}_{\beta,p})_{\infty}$, $1 \le p \le \infty$, was established in [1].

Remark 5. In Theorem 4, it was possible to improve the upper bound for the quantity $e_m(L^{\psi}_{\beta,p})_{\infty}$ obtained by A. S. Fedorenko:

$$\psi(m) m^{(\frac{1}{p}-\frac{1}{2})_+} \ll e_m(L^{\psi}_{\beta,p})_{\infty} \ll \psi(m) m^{(\frac{1}{p}-\frac{1}{2})_+} \sqrt{\log m}.$$

In the next assertion, we establish an exact order estimate for the quantity $e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty}$, $1 < p < \infty$.

Theorem 5. Let $1 < p < \infty$, $\psi \in B$, $\beta \in \mathbb{R}$ and, moreover, there exists $\varepsilon > 0$ such that the sequence $\psi(t) t^{\frac{1}{p}+\varepsilon}$, $t \in \mathbb{N}$, is non-increasing. Then the following estimate holds:

$$e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty} \asymp \psi(m) m^{\frac{1}{p}}.$$

Proof. We prove the upper bound. Let l and m be such that $2^l < m \le 2^{l+1}$. Successively using Minkowski's inequality, (18) and (20), we have

$$e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty} \ll \| f - \sum_{j<l} \delta_j(f) \|_{\infty} =$$

$$= \| \sum_{j \ge l} \delta_j(f) \|_{\infty} \le \sum_{j \ge l} \| \delta_j(f) \|_{\infty} \le \sum_{j \ge l} 2^{\frac{j}{p}} \| \delta_j(f) \|_p \ll$$

$$\ll \sum_{j \ge l} 2^{\frac{j}{p}} \psi(2^j) \| \delta_j(f^{\psi}_{\beta}) \|_p \ll \sum_{j \ge l} 2^{\frac{j}{p}} \psi(2^j). \qquad (29)$$

Taking into account that the sequence $\psi(t) t^{\frac{1}{p}+\varepsilon}$ is non-increasing, we continue the estimate (29):

$$e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty} \ll \sum_{j \ge l} 2^{j\left(\frac{1}{p}+\varepsilon\right)} \psi(2^j) 2^{-j\varepsilon} \ll$$

$$\ll 2^{l\left(\frac{1}{p}+\varepsilon\right)}\psi\left(2^l\right)\sum_{j\geq l}2^{-j\varepsilon} \ll \psi\left(2^l\right)2^{l\frac{1}{p}} \ll \psi(m)m^{\frac{1}{p}}.$$

The upper bound is proved.

To prove the lower bound, we use Theorem 2. Let Θ_m be an arbitrary set of m integers $n_1,...,n_m$, and for $f \in L^{\psi}_{\beta,p}$ let us set

$$S_{\grave{E}_m}(f,x) = \sum_{k=1}^{m} f(n_k)e^{in_k x}.$$

Then, taking into account (1), we can write

$$I_4 = \sup_{f\in L^{\psi}_{\beta,p}} \|f - S_{\Theta_m}(f)\|_{\infty} =$$

$$= \sup_{\|\varphi\|_p \leq 1} \| \varphi * F_{\psi} - \varphi * \sum_{k\in\Theta_m} e^{ikx} * F_{\psi} \|_{\infty}. \qquad (30)$$

Further, setting

$$t_m(x) = \sum_{k\in\Theta_m} e^{ikx} * F_{\psi}(x),$$

from (30) we will have

$$I_4 = \sup_{\|\varphi\|_p \leq 1} \| \varphi * (F_{\psi} - t_m) \|_{\infty} = \| F_{\psi} - t_m \|_{p'} \geq e^{\perp}_m(F_{\psi})_{p'},$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Hence, using Theorem 2, we arrive at the desired lower bound for the quantity $e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty}$.

Theorem 5 is proved.

Comparing the results of Theorems 4 and 5, we obtain that in the case $1 < p \leq 2$, the following relation holds between the quantities $e_m(L^{\psi}_{\beta,p})_{\infty}$ and $e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty}$:

$$e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty} \asymp m^{\frac{1}{2}} e_m(L^{\psi}_{\beta,p})_{\infty}.$$

If, however, $2 < p < \infty$, then the following relation holds:

Remark 6. In the case $\psi(|k|) = |k|^{-r}$, $r > \frac{1}{p}$, the order of the quantity $e^{\perp}_m(L^{\psi}_{\beta,p})_{\infty}$, $1 < p < \infty$ was established by A. S. Romanyuk.

CONCLUSIONS

The rapid development of modern computing technologies, machine learning methods, digital signal processing, and mathematical modeling of complex physical processes confirms that approximation theory is one of the most in-demand and strategically important branches of modern mathematics. It serves as a theoretical foundation for the discretization of continuous problems, minimization of computational costs, and development of highly efficient numerical algorithms, which makes it highly relevant for both fundamental science and applied engineering applications.

We have considered order estimates for the best approximations, approximations by Fourier sums, the best m-term approximations, and orthogonal trigonometric approximations of the classes $L^{\psi}_{\beta,p}$ in the space L_q under various relations between the parameters p and q.

Thus, the developed mathematical approaches and the established exact order estimates not only enrich the theoretical framework of functional analysis and approximation theory but also create a foundation for further research in the field of computational mathematics. They can be directly applied to the error analysis of numerical methods for solving integral and differential equations that describe periodic processes.

In the modern world, function approximation theory has expanded far beyond the boundaries of pure mathematics. It acts as the primary bridge between the analog world (where processes are continuous: sound, light, motion, time) and the digital world (where computers operate solely with discrete zeros and ones). Without the mathematical framework of this theory, the functioning of modern computing systems, artificial intelligence, and digital communications would be impossible.

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